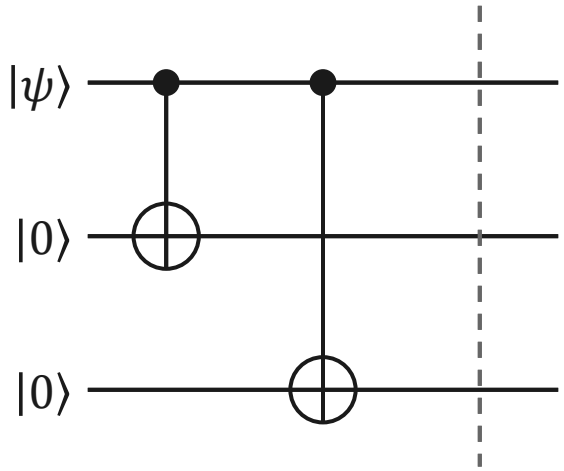
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Correcting Quantum Errors

Repetition, parity checks, the bit-flip, phase-flip and Shor codes

Chapter 8 · Lecture 17



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Today

1. **Repetition**
2. **Three obstacles**
3. **Bit-flip code**
4. **Phase-flip code**
5. **Shor's code**
6. **Discretisation**

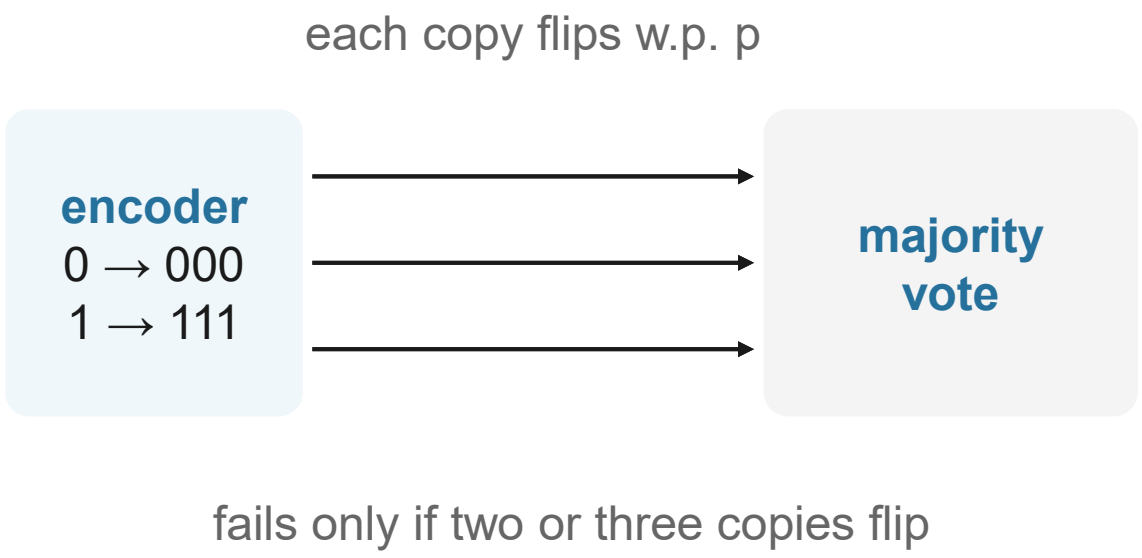
By the end of this lecture you can

- compute the logical error rate of a repetition code and say when coding helps
- explain why quantum codes measure parities, not qubits
- encode, extract a syndrome and correct one bit flip or one phase flip
- explain why correcting X and Z on every qubit corrects any single-qubit error

1 Redundancy without copying

When does sending a bit three times help?

The classical repetition code



Send each bit three times: $0 \to 000$, $1 \to 111$.

Each copy flips independently with probability p: the binary symmetric channel.

The receiver takes a majority vote; it fails only if two or three copies flipped.

YOUR TURN · PREDICT

Your turn: when does repetition help?

Think · 1 minute

The vote fails with probability p_L.

For which flip probabilities p is $p_L < p$?

Sketch p_L against p on [0, 1] with the line $p_L = p$.

ANSWER

The code helps exactly when p < ½

$p_L = 3p^2(1-p) + p^3$ (two flips, or three)

$p_L = 3p^2 - 2p^3$

$p_L < p \Leftrightarrow 3p - 2p^2 < 1 \Leftrightarrow (2p-1)(p-1) > 0$

$\Leftrightarrow p < \frac{1}{2}$ (for 0 < p < 1)

One flip is always corrected; only two coincident flips defeat the code.

How much does repetition gain?

p	$3p^2 - 2p^3$	sampled	p/p_L
0.01	0.000298	0.00030	34
0.1	0.028	0.02811	3.6
0.3	0.216	0.21590	1.4
0.5	0.5	0.50221	1

sampled: Simulation 8.1, 200 000 trials per row, seed 6581

Large gain for small p: 34× at p = 0.01.

Sampled rates agree within $\sqrt{p_L(1-p_L)/N} \approx$ 3.7e-04 at p = 0.1.

No gain at p = ½: the copies carry no information.

Parity checks: the syndrome

$$s = H\,r \pmod 2, \qquad H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

flipped bit	syndrome
none	00
r_0	10
r_1	11
r_2	01

Compute $r_0 \oplus r_1$ and $r_1 \oplus r_2$ instead of reading $r_0 r_1 r_2$.

Each single flip gives a different non-zero syndrome.

000 and 111 both give 00: the syndrome never reveals the message.

Distinct, non-zero columns of H: every single flip is correctable.

Three obstacles for qubits

1 No cloning

$\alpha|0\rangle + \beta|1\rangle$ cannot be copied (Chapter 4). Redundancy must come from entanglement.

2 Measurement disturbs

Reading a qubit destroys α, β. Measure only what does not depend on them.

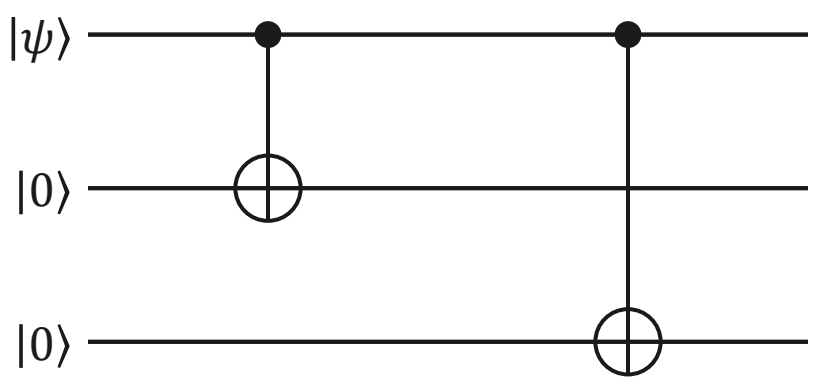
3 Errors are continuous

Any small rotation, dephasing or partial decay (Chapter 7). A table needs finitely many rows.

2 The three-qubit codes

How do we find a bit flip without looking at the qubits?

Encoding: entangle, do not copy



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\rightarrow \ \alpha|000\rangle + \beta|111\rangle = \alpha\left|\overline{0}\right\rangle + \beta\left|\overline{1}\right\rangle$$

Two CNOTs from qubit 0 give $\alpha|000\rangle + \beta|111\rangle$.

Not three copies: $(\alpha|0\rangle + \beta|1\rangle)^{\otimes 3}$ has cross terms such as $\alpha^2\beta|001\rangle$.

α and β live in the correlations, not in any one qubit.

Measure parities, not qubits

error	s_0	s_1
none	0	0
X on qubit 0	1	0
X on qubit 1	1	1
X on qubit 2	0	1

s_0 from Z_0Z_1, s_1 from Z_1Z_2

$$s = \frac{1}{2}(1 - \lambda), \qquad \lambda = \pm 1$$

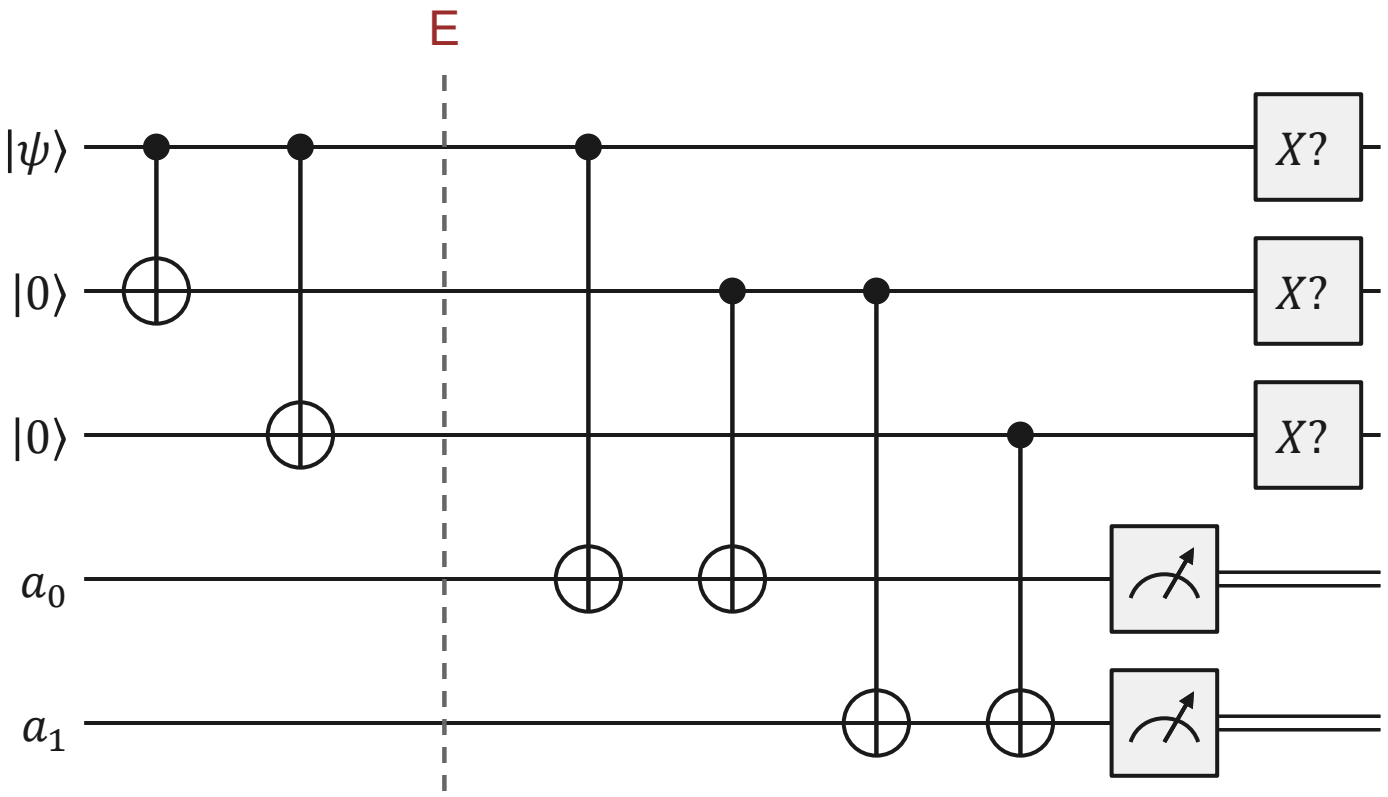
$Z_0Z_1 = Z \otimes Z \otimes I$: +1 if qubits 0 and 1 agree, −1 if they differ.

Both code words give +1 for both checks: the code state is left untouched.

After X_k the state is still an eigenstate; the eigenvalues name k.

Apply X_k again to undo it. α and β are never learned.

The syndrome circuit



Ancilla a_0 collects $q_0 \oplus q_1$; a_1 collects $q_1 \oplus q_2$.

Same parity in both branches: the ancilla ends in a definite state, unentangled with the data.

Measured $s_0 s_1$ = 10, 11, 01 selects X on qubit 0, 1, 2.

Deferred measurement (Chapter 3): Toffolis can replace the classical control.

Worked example: one bit flip, step by step

$$\alpha|010\rangle + \beta|101\rangle, \qquad |00\rangle_a$$

1 An X hits qubit 1: data α|010⟩ + β|101⟩, ancillas |00⟩.

Before the next slide

What does a_0 hold after its two CNOTs, in each branch?

Worked example: one bit flip, step by step

$$\alpha|010\rangle + \beta|101\rangle, \qquad |00\rangle_a$$

$$\rightarrow \ (\alpha|010\rangle + \beta|101\rangle) \otimes |11\rangle_a$$

1 An X hits qubit 1: data α|010⟩ + β|101⟩, ancillas |00⟩.

2 $q_0 \oplus q_1 = 1$ and $q_1 \oplus q_2 = 1$ in both branches: a product of data and ancillas.

Before the next slide

What does measuring the ancillas tell us about α and β?

Worked example: one bit flip, step by step

$$\alpha|010\rangle + \beta|101\rangle, \qquad |00\rangle_a$$

$$\rightarrow \ (\alpha|010\rangle + \beta|101\rangle) \otimes |11\rangle_a$$

$$s_0 s_1 = 11 \ \rightarrow \ X_1\colon \alpha|000\rangle + \beta|111\rangle$$

1 An X hits qubit 1: data $\alpha|010\rangle + \beta|101\rangle$, ancillas $|00\rangle$.

2 $q_0 \oplus q_1 = 1$ and $q_1 \oplus q_2 = 1$ in both branches: a product of data and ancillas.

3 Syndrome 11 with certainty, for every α, β; X_1 restores the code state.

YOUR TURN · CALCULATE

Your turn: two bit flips

Think · 1 minute

An error X_0X_1 hits $\alpha|000\rangle + \beta|111\rangle$.

Which syndrome is measured?

What does the decoder apply, and what is the final state?

ANSWER

Syndrome 01: the decoder makes it worse

Corrupted state $\alpha|110\rangle + \beta|001\rangle$: syndrome $s_0 s_1 = 01$.

The decoder applies X_2: net $X_0 X_1 X_2 = \overline{X}$, which swaps $|\overline{0}\rangle$ and $|\overline{1}\rangle$.

Failure with two or three flips: $p_L = 3p^2 - 2p^3$ again.

The syndrome of two flips looks like a different single flip; the code fails silently.

COMMON MISTAKE

“The three-qubit code protects against all errors”

✗ A Z error passes unnoticed

$$Z_k\colon\ \alpha|000\rangle + \beta|111\rangle$$

$$\rightarrow\ \alpha|000\rangle - \beta|111\rangle$$

Both parities stay even: syndrome 00, yet the state has changed.

✓ It corrects one X, nothing else

X errors: syndromes 10, 11, 01

Z errors: syndrome 00

Both kinds need a larger code

Shor's code (part three) protects against both kinds.

The bit-flip code corrects one X error; a single Z error becomes a logical phase error.

DERIVATION

The phase-flip code: rotate the basis

$HZH = X$

a Hadamard exchanges the two kinds of error

$|\overline{0}\rangle = |{+}{+}{+}\rangle, \qquad |\overline{1}\rangle = |{-}{-}{-}\rangle$

encode as before, then H on every qubit

Z_k: $|+\rangle \rightarrow |-\rangle$ on qubit k

a phase flip is a bit flip in the ± basis

checks: X_0X_1, X_1X_2

+1 when two qubits agree in the ± basis

Corrects one Z error and is blind to X: the mirror image of the bit-flip code.

SIMULATION

Simulation: the bit-flip code with ancillas

```
qc.cx(0, 1); qc.cx(0, 2)
qc.x(err)
qc.cx(0, 3); qc.cx(1, 3)
qc.cx(1, 4); qc.cx(2, 4)
qc.measure([3, 4], [0, 1])
for v, q in ((1, 0), (3, 1), (2, 2)):
    with qc.if_test((qc.cregs[0], v)):
        qc.x(q)
```

Syndromes printed in textbook order s_0 s_1 check.

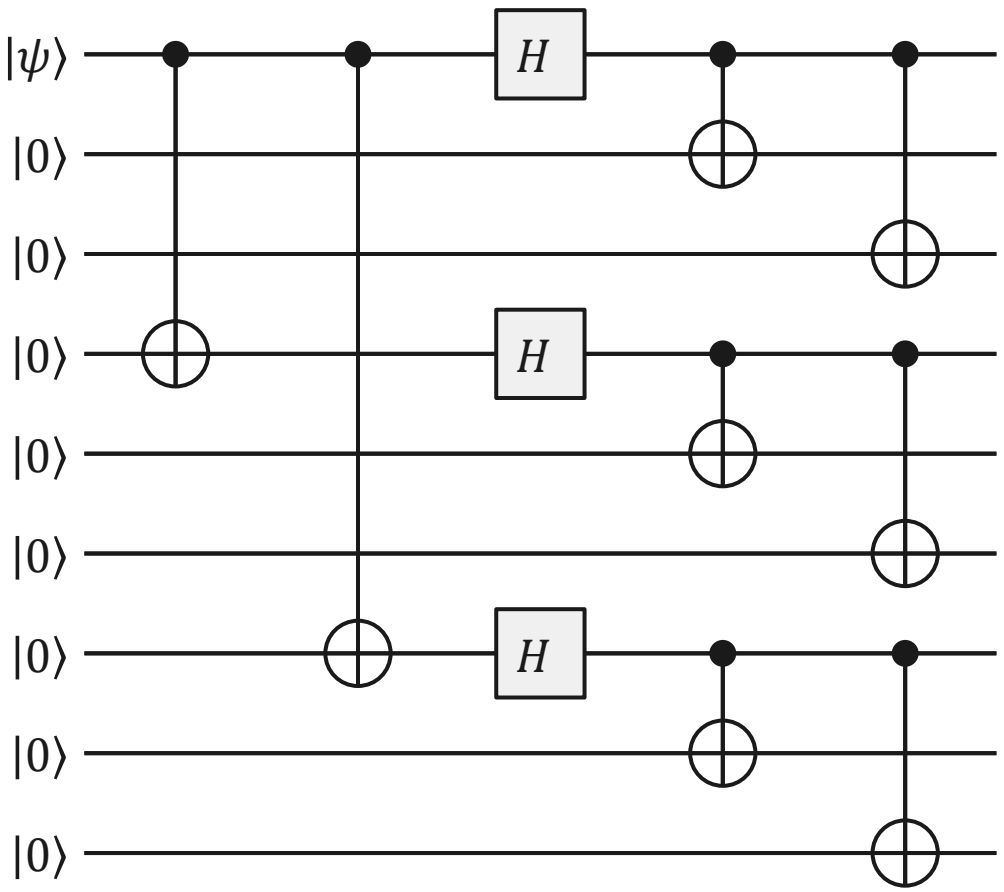
Qiskit reads the register as $v = s_0 + 2s_1$.

```
none: F = 1.000, 000×1000
X q0: F = 1.000, 100×1000
X q1: F = 1.000, 110×1000
X q2: F = 1.000, 010×1000
```

3 Shor's code and discrete errors

Can one code correct every single-qubit error?

Shor's code: two levels



Outer level: the phase-flip code on qubits 0, 3, 6.

Inner level: each of them gets its own bit-flip code (blocks 0–2, 3–5, 6–8).

$$|\overline{0}\rangle = \frac{(|000\rangle + |111\rangle)^{\otimes 3}}{2\sqrt{2}},$$

$$|\overline{1}\rangle = \frac{(|000\rangle - |111\rangle)^{\otimes 3}}{2\sqrt{2}}$$

A code built from codes: concatenation.

Syndromes of the Shor code

six Z-checks (bit flips, per block)

$$Z_0Z_1, Z_1Z_2, Z_3Z_4,$$

$$Z_4Z_5, Z_6Z_7, Z_7Z_8$$

two X-checks (block signs)

$$X_0X_1X_2X_3X_4X_5$$

$$X_3X_4X_5X_6X_7X_8$$

An X inside a block: that block's two Z-parities.

A Z on any qubit flips its block's sign: $|000\rangle + |111\rangle \rightarrow |000\rangle - |111\rangle$.

$X \otimes X \otimes X$ reads a block's sign; the two X-checks compare blocks: 10, 11, 01.

$Y = iXZ$ triggers both kinds of check and gets both corrections.

YOUR TURN · CALCULATE

Your turn: two errors in the Shor code

Pairs · 90 seconds

Find the syndrome and the correction for

(a) Z_4, a phase flip on qubit 4;

(b) Y_0, a Y error on qubit 0.

ANSWER

Z_4: block 1; Y_0: both levels

(a) Z_4 commutes with every Z-check; anticommutes with both X-checks: outer syndrome 11, block 1.

Apply Z_3 (or Z_4, Z_5): Z_3Z_4 is a check, the identity on the code.

(b) Y_0: inner 10 $\rightarrow$ X_0; outer 10 $\rightarrow$ Z on block 0, say Z_0.

$Z_0X_0 = iY_0$: the error is undone up to a global phase.

A correction need only be equivalent to the error, not equal to it.

DERIVATION

Continuous errors become discrete

$$E = e_I\, I + e_X\, X + e_Y\, Y + e_Z\, Z, \qquad e_\sigma = \frac{1}{2}\mathrm{Tr}(\sigma\, E)$$

any one-qubit operator; the Paulis are orthogonal

$$E_k|\psi_L\rangle = \sum_\sigma e_\sigma\, \sigma_k|\psi_L\rangle$$

four terms with four different syndromes

$$\Pr(\sigma) = \frac{|e_\sigma|^2}{\sum_\tau |e_\tau|^2}$$

measuring the checks keeps one term

$$\sigma_k\, \sigma_k|\psi_L\rangle = |\psi_L\rangle$$

a Pauli correction, whatever the outcome

Measuring the checks picks one of four discrete errors: Pauli corrections suffice.

SIMULATION

Simulation: a random error in the Shor code

```
U = random_unitary(2, seed=6583).data
e2 = {k: abs(np.trace(s @ U) / 2) ** 2
for s in itertools.product((0, 1), repeat=8):
    phi = (phi + (-1) ** si * (g @ phi)) / 2
fid += abs(np.vdot(logical, phi)) ** 2
```

Projects onto each of the 2^8 syndrome patterns, corrects, and adds up the fidelity.

```
|e|²:  I 0.1560  X 0.3531
       Y 0.0084  Z 0.4825
q0, q4, q8: Pr(class) = |e|²
fidelity 1.000000000 (×3)
```

COMMON MISTAKE

“Continuous errors need continuous corrections”

✗ Undo the measured rotation

rotation by 0.1 rad?
measure the angle,
rotate back by −0.1 rad

The syndrome never reveals the angle.

✓ Collapse, then a Pauli

measure the checks:
mostly I, sometimes X, Y, Z
apply that Pauli

Only Pauli corrections are ever applied.

Correcting X, Y and Z on every qubit is enough: the measurement makes the error discrete.

Recap

1

Repetition

$p_L = 3p^2 - 2p^3 < p$ exactly when $p < \frac{1}{2}$

2

Parities, not qubits

Z_0Z_1, Z_1Z_2 locate a flip; α, β never revealed

3

Two codes, then Shor

bit flip, phase flip (H); nine qubits: any single error

4

Discretisation

measurement collapses E onto I, X, Y or Z

After the break: Lecture 18

Stabilisers, the surface code and fault tolerance

Read: handout 8.1–8.3 before next week; 8.8 is self-study.

Exit ticket

Shor code: give the syndrome and the correction for an X error on qubit 7.

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