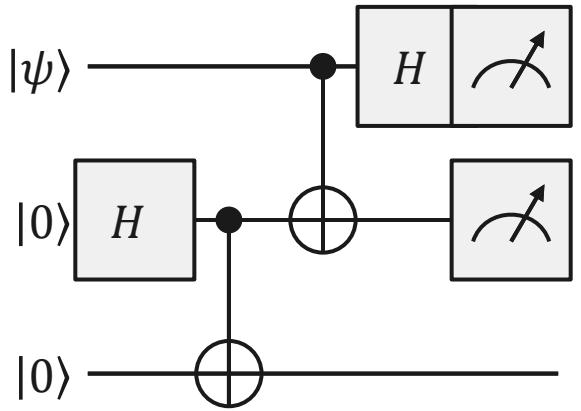
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Teleportation, Superdense Coding and CHSH



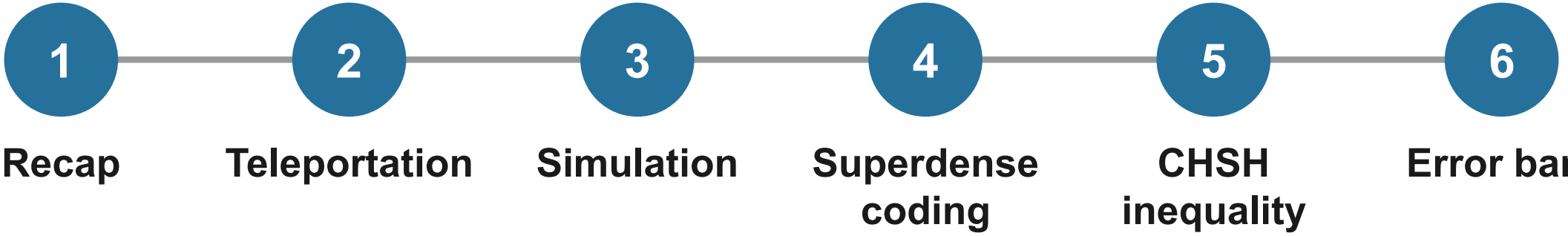
What a shared Bell pair can do, and what no classical model can

Chapter 4 · Lecture 9

Sajid Muhaimin Choudhury

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Today



By the end of this lecture you can

- derive teleportation, with the correction for each of the four outcomes, and state its cost
- derive superdense coding and its cost
- derive the classical CHSH bound and the quantum value $2\sqrt{2}$ for $|\Phi^+\rangle$
- estimate S with its standard error from simulated shots

YOUR TURN · RECALL

Recap: reading a Bell state

Recall · 30 seconds

The Bell measurement (CNOT, then H on qubit 0) is applied to $|\Psi^+\rangle$. Which outcome $m_0 m_1$ is obtained?

A 00

B 01

C 10

D 11

ANSWER

B: $|\Psi^+\rangle$ gives 01

$\mathrm{CNOT}_{01}|\Psi^+\rangle = |+\rangle|1\rangle, \qquad H \otimes I\colon |01\rangle$

$|\Phi^+\rangle \to 00, |\Psi^+\rangle \to 01, |\Phi^-\rangle \to 10, |\Psi^-\rangle \to 11$

Today Alice and Bob use this table to send states and bits.

1 Quantum teleportation

How can Alice give Bob a qubit she cannot measure or copy?

The task

Alice

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

qubit 1 of the pair

Bob

qubit 2 of the pair

shared in advance: $|\Phi^+\rangle$ on qubits 1 and 2

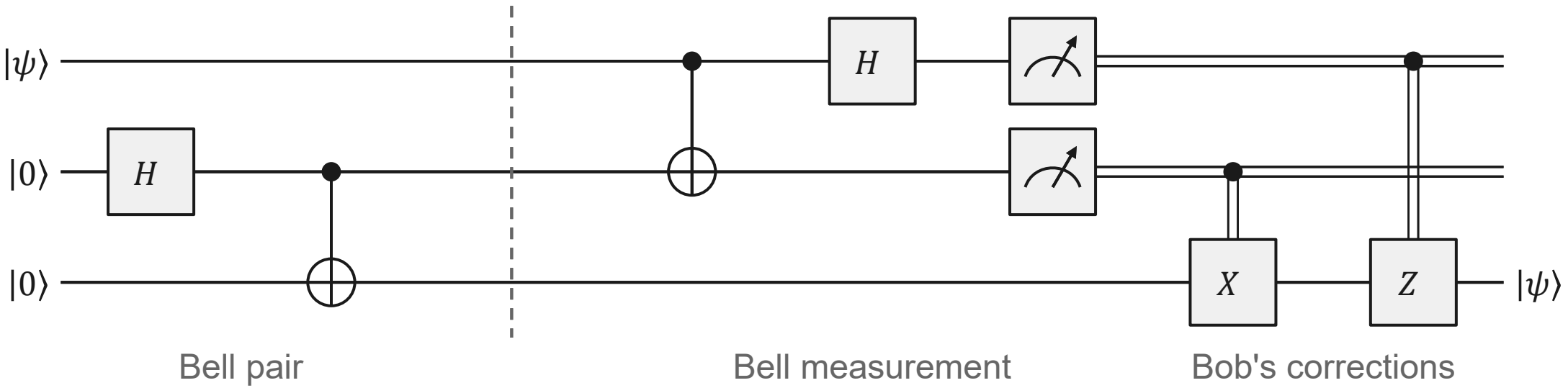
Alice does not know α and β.

Measuring gives one bit, not α and β.

She cannot copy the state (no-cloning).

She has: half of a shared $|\Phi^+\rangle$ and a classical channel.

The teleportation circuit



1. Qubits 1 and 2 become $|\Phi^+\rangle$; Bob takes qubit 2.
2. Alice makes a Bell measurement on qubits 0 and 1: bits $m_0 m_1$.
3. She sends the two bits to Bob over a classical channel (double wires).
4. Bob applies X^{m_1}, then Z^{m_0}: his qubit is $|\psi\rangle$.

DERIVATION

Why it works: regroup the state

$$|\psi\rangle|\Phi^+\rangle = (\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)/\sqrt{2}$$ slice 1

$$|00\rangle, |11\rangle = (|\Phi^+\rangle \pm |\Phi^-\rangle)/\sqrt{2}$$ qubits 0, 1 in the Bell basis

$$|01\rangle, |10\rangle = (|\Psi^+\rangle \pm |\Psi^-\rangle)/\sqrt{2}$$

$$= ½(|\Phi^+\rangle|\psi\rangle + |\Psi^+\rangle X|\psi\rangle + |\Phi^-\rangle Z|\psi\rangle + |\Psi^-\rangle XZ|\psi\rangle)$$ collect terms

$$X|\psi\rangle = \beta|0\rangle + \alpha|1\rangle, \qquad Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$$ Bob's qubit

Nothing has happened yet: the same state, written in the Bell basis of Alice's qubits.

YOUR TURN · PREDICT

Your turn: Alice obtains 10

Think · 45 seconds

Alice's Bell measurement gives $m_0 m_1 = 10$. What does Bob hold, and what must he do?

A $|\psi\rangle$; nothing

B $X|\psi\rangle$; apply X

C $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$; apply Z

D $XZ|\psi\rangle$; apply X, then Z

ANSWER

C: Bob holds Z|ψ⟩ and applies Z

Outcome 10 means Alice's qubits are in $|\Phi^-\rangle$.

It multiplies $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$; since $Z^2 = I$, Bob applies Z.

All four branches

outcome	Bob before	correction	after
00	$\alpha\|0\rangle + \beta\|1\rangle$	none	$\|\psi\rangle$
01	$\beta\|0\rangle + \alpha\|1\rangle$	X	$\|\psi\rangle$
10	$\alpha\|0\rangle - \beta\|1\rangle$	Z	$\|\psi\rangle$
11	$-\beta\|0\rangle + \alpha\|1\rangle$	X, then Z	$\|\psi\rangle$

Each outcome has probability ¼, whatever α and β are; Bob's correction is X^{m_1} then Z^{m_0}:

$$Z^{m_0} X^{m_1} \, X^{m_1} Z^{m_0} |\psi\rangle = |\psi\rangle$$

DERIVATION

Worked example: a complex amplitude

Alice teleports $|\psi\rangle = (3|0\rangle + 4i|1\rangle)/5$. Each outcome has probability ¼.

00: $(3|0\rangle + 4i|1\rangle)/5$ no correction

01: $(4i|0\rangle + 3|1\rangle)/5$ X swaps the amplitudes

10: $(3|0\rangle - 4i|1\rangle)/5$ Z restores the sign

11: $(-4i|0\rangle + 3|1\rangle)/5$ X, then Z

Without the bits: Pr(Bob sees 0) = ¼(9 + 16 + 9 + 16)/25 = ½, which says nothing about |ψ⟩.

What teleportation does not do

No copy: Alice's qubits end in $|m_0 m_1\rangle$; $|\psi\rangle$ exists only at Bob's.

No signal: until the bits arrive, Bob's statistics do not depend on $|\psi\rangle$.

No matter moves: the state moves, not the particle.

The ebit is used up: afterwards Bob's qubit is not entangled with Alice's.

teleportation

1 ebit + 2 bits → 1 qubit

superdense coding (part 2)

1 ebit + 1 qubit → 2 bits

COMMON MISTAKE

“One classical bit is enough”

✗ Send only one bit

Bob gets m_1 and applies X if needed

the state is then right

Bob cannot tell 00 (no correction) from 10 (Z).

✓ Four outcomes need two bits

four equally likely outcomes

four different corrections

without m_0: $Z|\psi\rangle$ half the time

With one ebit, fewer than two bits cannot teleport a qubit.

Teleportation needs both bits: m_1 decides the X and m_0 decides the Z.

SIMULATION

Simulation 4.4: teleportation with if_test

```
    qc.h(1)
    qc.cx(1, 2)
    qc.cx(0, 1)
    qc.h(0)
    qc.measure([0, 1], [0, 1])
    with qc.if_test((qc.clbits[1], 1)):
        qc.x(2)
    with qc.if_test((qc.clbits[0], 1)):
        qc.z(2)
    qc.rz(-B, 2)
    qc.ry(-A, 2)
    qc.measure(2, 2)
```

if_test: gates controlled by a classical bit.

Output condensed from teleportation.out (textbook order).

$$|\psi\rangle = R_z(0.7)\, R_y(1.1)|0\rangle$$

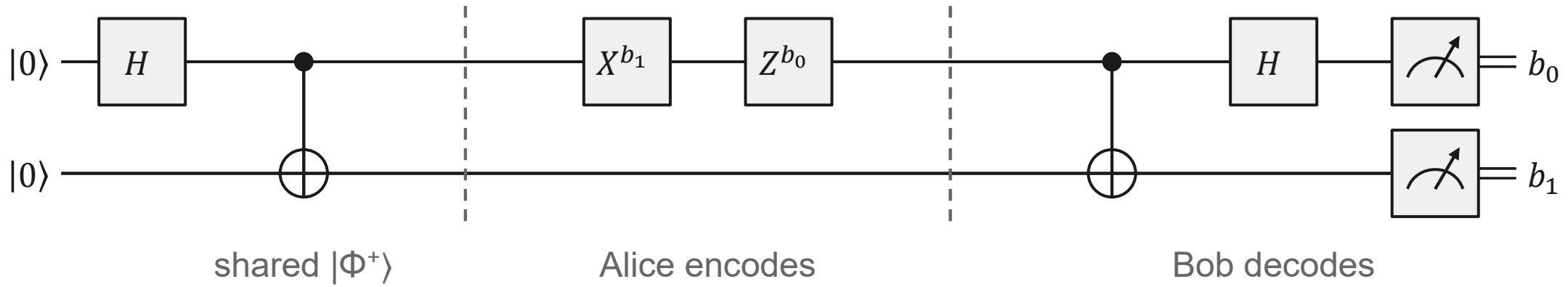
Bob undoes the preparation:
the check bit must always be 0.

```
Alice 00: 525  01: 477
      10: 502  11: 496
check bit 0: 2000 of 2000
deferred: 1.000000000000
```

2 Superdense coding

Can one qubit carry two bits?

The superdense-coding circuit



1. Alice and Bob share $|\Phi^+\rangle$ before the message exists.
2. Alice encodes $b_0 b_1$: X^{b_1}, then Z^{b_0} on her qubit.
3. She sends her one qubit to Bob.
4. Bob, holding both qubits, makes a Bell measurement and reads two bits.

YOUR TURN · PREDICT

Your turn: sending 11

Think · 30 seconds

Alice wants to send 11. Starting from $|\Phi^+\rangle$, which Bell state does Bob receive?

A $|\Phi^+\rangle$

B $|\Phi^-\rangle$

C $|\Psi^+\rangle$

D $|\Psi^-\rangle$

ANSWER

D: $|\Psi^-\rangle$, which Bob reads as 11

X on qubit 0: $(|10\rangle + |01\rangle)/\sqrt{2} = |\Psi^+\rangle$

Z on qubit 0: $(|01\rangle - |10\rangle)/\sqrt{2} = |\Psi^-\rangle$

Bob: CNOT gives $|-\rangle|1\rangle$, H gives $|11\rangle$.

DERIVATION

The encoding table

00: $\lvert\Phi^+\rangle$	do nothing
01: $X \otimes I \lvert\Phi^+\rangle = \lvert\Psi^+\rangle$	flip the first labels
10: $Z \otimes I \lvert\Phi^+\rangle = \lvert\Phi^-\rangle$	sign of the $\lvert 11\rangle$ term
11: $ZX \otimes I \lvert\Phi^+\rangle = \lvert\Psi^-\rangle$	both

The Bell measurement returns $b_0 b_1$ in the order it went in.

Two bits per qubit?

A modem symbol from M waveforms carries $\log_2 M$ bits; a qubit looks like a one-bit symbol.

Without entanglement, one qubit carries at most one bit; with one ebit, at most two.

The extra bit is paid for: the ebit had to be distributed, which itself sends a qubit.

End to end: two qubits travel to carry two bits.

teleportation

1 ebit + 2 bits → 1 qubit

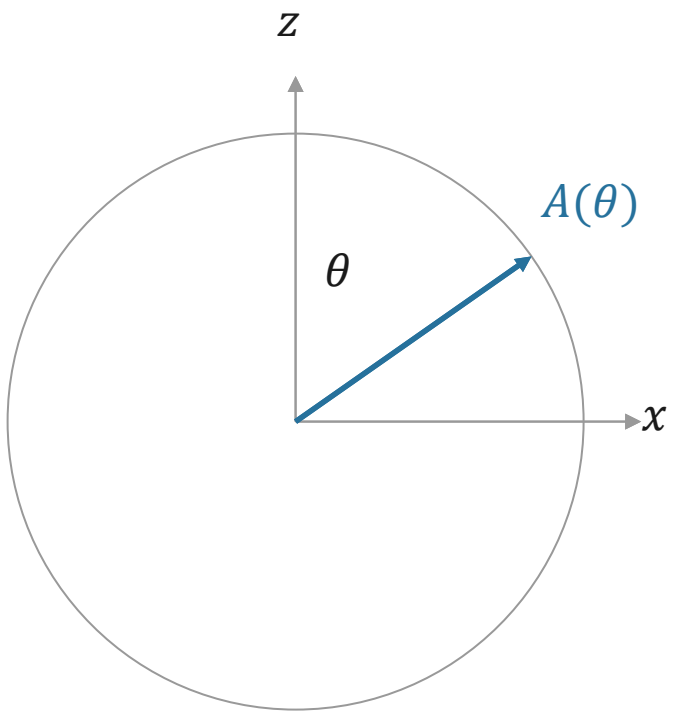
superdense coding (part 2)

1 ebit + 1 qubit → 2 bits

3 The CHSH inequality

Could the outcomes be answers written down in advance?

The CHSH set-up



Alice and Bob each choose one of two settings at random and record ±1.

$A(\theta) = \cos\theta\, Z + \sin\theta\, X$: measure with $R_y(-\theta)$, then a meter; 0 is +1, 1 is −1.

Correlator: $E(\theta_A, \theta_B) = \Pr(\text{same}) - \Pr(\text{different})$

$$S = E(a_0, b_0) + E(a_0, b_1) + E(a_1, b_0) - E(a_1, b_1)$$

DERIVATION

The classical bound |S| ≤ 2

Assume predetermined values $\alpha_0, \alpha_1, \beta_0, \beta_1 = \pm 1$.

No influence of one side's setting on the other; free choice of settings.

$$\alpha_0\beta_0 + \alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_1\beta_1 = \alpha_0(\beta_0 + \beta_1) + \alpha_1(\beta_0 - \beta_1)$$ regroup one pair

$\beta_0, \beta_1 = \pm 1$: one bracket is 0, the other ± 2

each pair contributes ± 2

$S =$ average over pairs $\Rightarrow |S| \leq 2$ any distribution

Local hidden-variable models obey |S| ≤ 2 (the CHSH inequality).

DERIVATION

The quantum correlator for $|\Phi^+\rangle$

$$A(\theta_A) \otimes A(\theta_B) = c_A c_B\, ZZ + c_A s_B\, ZX + s_A c_B\, XZ + s_A s_B\, XX$$

c = cos, s = sin

$$ZZ|\Phi^+\rangle = XX|\Phi^+\rangle = |\Phi^+\rangle$$

these terms give 1

$$ZX|\Phi^+\rangle = |\Psi^-\rangle, \qquad XZ|\Phi^+\rangle = -|\Psi^-\rangle$$

orthogonal: they give 0

$$E(\theta_A, \theta_B) = \cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B = \cos(\theta_A - \theta_B)$$

The correlation depends only on the angle between the two directions.

YOUR TURN · PREDICT

Predict, then simulate: the value of S

Think · 45 seconds

Settings $a_0 = 0, a_1 = \pi/2, b_0 = \pi/4, b_1 = -\pi/4$. Using $E = \cos(\theta_A - \theta_B)$, what is S?

A 2

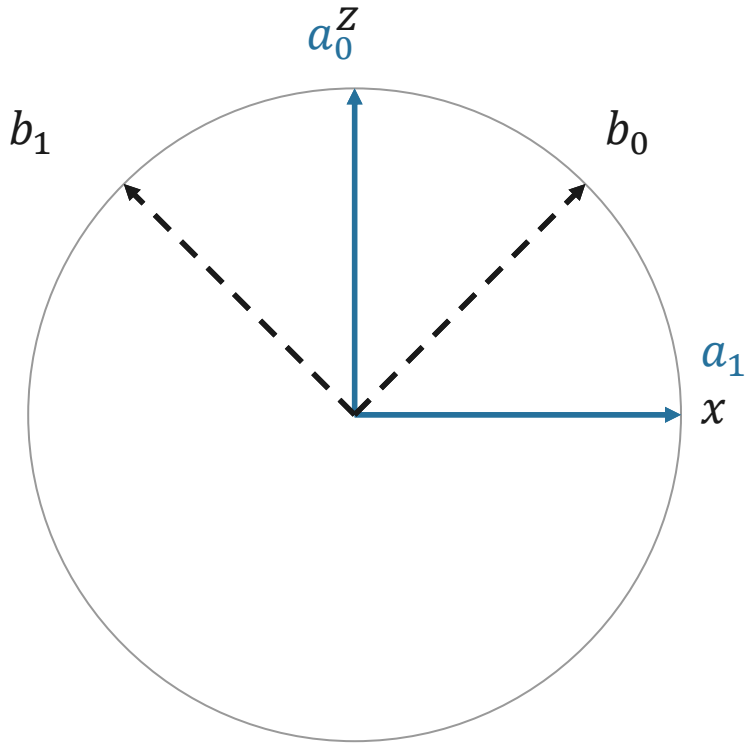
B $\sqrt{2}$

C $2\sqrt{2}$

D 4

ANSWER

C: S = 2√2 ≈ 2.83 > 2



Alice solid, Bob dashed; three pairs π/4 apart, one 3π/4

$$S = 3\cos\frac{\pi}{4} - \cos\frac{3\pi}{4} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Tsirelson: for any quantum state and ±1 observables, $|S| \leq 2\sqrt{2}$.

Polarisers: angles halve, so Bob's are at ±22.5°, Alice's at 0° and 45°.

Simulation 4.6: S from simulated shots

E(a0, b0)	+0.6825	E(a0, b1)	+0.7225
E(a1, b0)	+0.7095	E(a1, b1)	-0.6935
exact S	2.8284	best deterministic |S|	2

sampled S = 2.8080 ± 0.0225 (4000 shots per setting)

Standard error of one correlator: $\sqrt{(1-E^2)/N}$; of S: $\sigma_S = \sqrt{2/N}$.

Five standard errors above 2 needs $N \geq 73$ shots per setting.

COMMON MISTAKE

“E is the probability of agreeing”

✗ Mix up E and probability

for settings π/4 apart: $E = \cos^2(\pi/8)$

and S can reach 4

Confuses a correlator with a probability, and Bloch with polariser angles.

✓ Keep the quantities apart

$E = \cos(\pi/4) \approx 0.707$

$\Pr(\text{same}) = (1 + E)/2 = \cos^2(\pi/8) \approx 0.854$

$|S| \leq 2\sqrt{2}$ (Tsirelson), not 4

E lies in [−1, 1]; probabilities lie in [0, 1].

E = Pr(same) − Pr(different); Pr(same) = (1 + E)/2. Do not halve Bloch angles; quantum S stops at 2√2.

Recap

1

Teleportation

1 ebit + 2 bits → 1 qubit; Bob applies X^{m_1} then Z^{m_0}

2

Superdense coding

1 ebit + 1 qubit → 2 bits; encode with $Z^{b_0}X^{b_1}$

3

Nothing for free

no copying, no faster-than-light signal; the ebit is consumed

4

CHSH

predetermined answers: $|S| \le 2$; $|\Phi^+\rangle$: $2\sqrt{2}$, the quantum maximum

After the break: Lecture 10 (Chapter 5)

Oracles and phase kickback: Deutsch, Deutsch–Jozsa, Bernstein–Vazirani

Read handout 4.6–4.8; run Simulation 4.5 at home.

Exit ticket

Alice teleports $|-\rangle$ and obtains 11. What does Bob hold before his corrections? Check that X, then Z, gives $|-\rangle$.

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