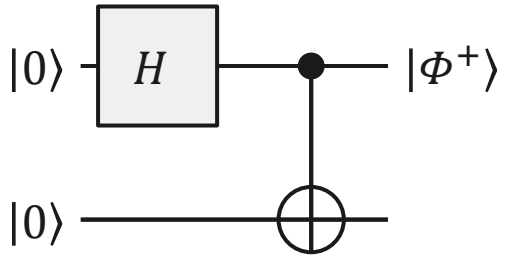
SAJID Lab
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EEE 6516 · QUANTUM COMPUTING

Entanglement and the Bell States

Correlation, no-signalling, no-cloning and the Bell measurement

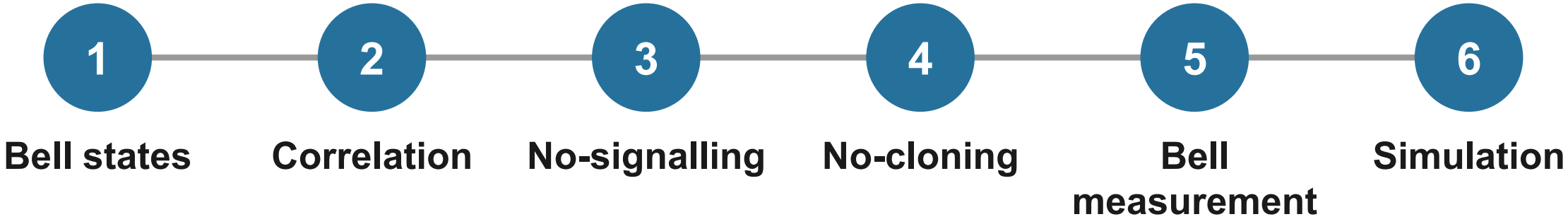


Chapter 4 · Lecture 8

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Today



By the end of this lecture you can

- write the four Bell states, prepare each with H and CNOT, and show that each is entangled
- tell entanglement from classical correlation using two measurement bases
- show that measuring half of a Bell pair sends no signal
- prove the no-cloning theorem; design and analyse the Bell measurement

YOUR TURN · RECALL

Recap: is this a product state?

Recall · 30 seconds

Is $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ a product state? Use the test $a_{00}a_{11} - a_{01}a_{10}$.

A Yes: the test gives 0

B No: the test gives $-\frac{1}{2}$

C No: the test gives $+\frac{1}{2}$

D Cannot tell without measuring

ANSWER

B: the test gives −½, so the state is entangled

$$a_{00} = a_{11} = 0, \qquad a_{01} = a_{10} = \frac{1}{\sqrt{2}}$$

$$a_{00}a_{11} - a_{01}a_{10} = 0 - \frac{1}{2} = -\frac{1}{2} \neq 0$$

This state has a name: it is one of today's four Bell states.

1 The Bell states

Which four entangled states form a basis?

The four Bell states

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \qquad |\Phi^-\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$

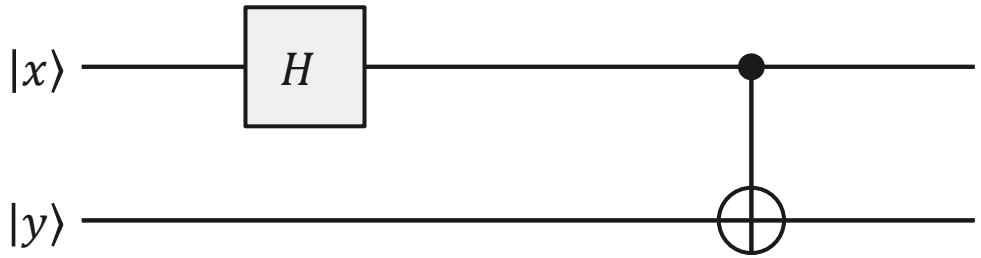
$$|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \qquad |\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

Letter: in Φ the two qubits agree, in Ψ they disagree.

Sign: a relative phase between the two terms.

Orthonormal: the **Bell basis**; one pair shared by two parties is an **ebit**.

Preparing a Bell state



H on qubit 0, then CNOT from qubit 0 to qubit 1

Start from $|xy\rangle$. After H on qubit 0:

$$\frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)|y\rangle$$

The CNOT flips y in the second term:

$$\frac{1}{\sqrt{2}}(|0\rangle|y\rangle + (-1)^x|1\rangle|\bar{y}\rangle)$$

y picks Φ or Ψ; x picks the sign.

YOUR TURN · PREDICT

Predict: the output for $|11\rangle$

Think · 45 seconds

Apply H on qubit 0 and then CNOT_{01} to $|11\rangle$. Which Bell state comes out?

A $|\Phi^+\rangle$

B $|\Phi^-\rangle$

C $|\Psi^+\rangle$

D $|\Psi^-\rangle$

ANSWER

D: |11⟩ gives |Ψ⁻⟩

$$|00\rangle \rightarrow |\Phi^+\rangle$$

$$|01\rangle \rightarrow |\Psi^+\rangle$$

$$|10\rangle \rightarrow |\Phi^-\rangle$$

$$|11\rangle \rightarrow |\Psi^-\rangle$$

x = 1 gives the minus sign; y = 1 gives a Ψ state:

$$\frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) = |\Psi^-\rangle$$

Ψ⁺ comes from |01⟩: the minus sign comes from H acting on |1⟩.

COMMON MISTAKE

“The sign in $|\Phi^-\rangle$ is a global phase”

✗ A sign is just a phase

$|\Phi^-\rangle$ and $|\Phi^+\rangle$ differ by a sign

so they are the same physical state

Confuses a relative phase with a global phase.

✓ The sign is on one term only

$$\langle\Phi^+|\Phi^-\rangle = \frac{1}{2}(1-1) = 0$$

$-|\Phi^+\rangle$ is the same state as $|\Phi^+\rangle$

Orthogonal states are perfectly distinguishable.

A relative phase changes the state; a global phase multiplies every term and changes nothing.

2 Entanglement is not just correlation

Can a pair of coins imitate $|\Phi^+\rangle$?

Two pairs that agree

Entangled pair

$$|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

Classically correlated pair

toss a fair coin; prepare $|00\rangle$ on heads, $|11\rangle$ on tails

Measure both in the computational basis:

both pairs give 00 or 11, each with probability ½;

the two qubits always agree, in both cases.

One basis cannot tell the pairs apart.

DERIVATION

$|\Phi^+\rangle$ in the X basis

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \qquad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

X-basis states (Chapter 2)

$$|\Phi^+\rangle$$
$$= \frac{1}{2\sqrt{2}}((|+\rangle + |-\rangle)(|+\rangle + |-\rangle) + (|+\rangle - |-\rangle)(|+\rangle - |-\rangle))$$

substitute into both terms

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|+\,+\rangle + |-\,-\rangle)$$

the cross terms cancel

In the X basis $|\Phi^+\rangle$ is again perfectly correlated: both + or both −.

Two bases, two pairs

$|\Phi^+\rangle$, Z basis

	0	1
0	$\frac{1}{2}$	0
1	0	$\frac{1}{2}$

coin pair, Z basis

	0	1
0	$\frac{1}{2}$	0
1	0	$\frac{1}{2}$

$|\Phi^+\rangle$, X basis

	+	−
+	$\frac{1}{2}$	0
−	0	$\frac{1}{2}$

coin pair, X basis

	+	−
+	$\frac{1}{4}$	$\frac{1}{4}$
−	$\frac{1}{4}$	$\frac{1}{4}$

Rows: qubit 0; columns: qubit 1.

Z basis: the two tables are identical.

X basis: only the entangled pair stays correlated.

Marginals are ½ everywhere; the joint table differs.

Simulation 4.1: 2000 shots of each pair

entangled, Z basis	`00 1072, 11 928`
entangled, X basis	`00 995, 11 1005`
classical, Z basis	`00 1041, 11 959`
classical, X basis	`00 522, 01 519, 10 472, 11 487`

The coin toss is a mid-circuit measurement of qubit 0 before the CNOT; counts in textbook order.

Binomial σ ≈ 22 for 1000, ≈ 19 for 500.

COMMON MISTAKE

“Correlated outcomes mean entanglement”

✗ One basis is enough

the qubits always agree in Z

so the pair is entangled

A coin and two copies of its outcome agree just as well.

✓ Test two bases (or CHSH)

agreement in Z and in X

no unentangled pair does both

Lecture 9: the CHSH test

Evidence of entanglement needs more than one basis.

Perfect agreement in one basis is produced by a coin; entanglement shows up across bases.

3 Measuring half a pair

Can Alice send Bob a message by measuring?

Alice measures her half

Alice
qubit 0

Bob
qubit 1, far away

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|+\,+\rangle + |-\,-\rangle)$$

Alice measures Z: 0 or 1, each ½; Bob's qubit is left in $|0\rangle$ or $|1\rangle$.

Alice measures X: + or −, each ½; Bob's qubit is left in $|+\rangle$ or $|-\rangle$.

Alice's choice decides which kind of state Bob holds. Can Bob notice?

DERIVATION

Bob's statistics do not change

Bob measures Z. Probability that he sees 0:

Alice measures Z: $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$ — Bob holds $|0\rangle$ or $|1\rangle$

Alice measures X: $\frac{1}{2}|\langle 0| +\rangle|^2 + \frac{1}{2}|\langle 0| -\rangle|^2 = \frac{1}{2}$ — Bob holds $|+\rangle$ or $|-\rangle$

Alice does nothing: $|a_{00}|^2 + |a_{10}|^2 = \frac{1}{2}$ — directly from $|\Phi^+\rangle$

No-signalling: nothing Alice does locally changes Bob's statistics.

YOUR TURN · CHECKPOINT

Checkpoint: after Alice's outcome

Think · 40 seconds

Alice measures her half of $|\Phi^+\rangle$ in the computational basis and obtains 1.

(a) What is the state of Bob's qubit?

(b) Before Alice tells him, what is the probability that Bob obtains 1?

COMMON MISTAKE

“Alice's measurement sends Bob a message”

✗ Alice chooses, Bob's state changes

Z or X decides Bob's state

so Bob receives a bit

Bob cannot detect which state he holds.

✓ Statistics, not states, reach Bob

Bob's outcome probabilities are unchanged

the lists are correlated only when compared

comparing needs a classical channel

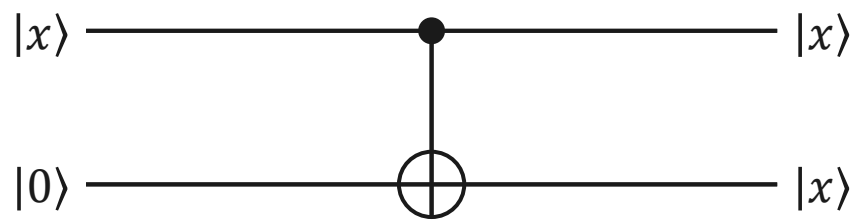
Correlations appear only when the two lists meet.

A measurement changes the state assigned to the other qubit, but not anything the other party can see.

4 No-cloning

Can a quantum state be copied like a bit?

A quantum fan-out?



x ∈ {0, 1}: the CNOT copies basis states

Classical fan-out copies a voltage to many gates.

Copies of an unknown state would let us estimate its amplitudes to any accuracy …

… and let Bob read Alice's basis, breaking no-signalling.

The CNOT copies basis states: $|x\rangle|0\rangle \to |x\rangle|x\rangle$.

YOUR TURN · PREDICT

Predict: the CNOT copier on $|+\rangle$

Think · 45 seconds

What is $\mathrm{CNOT}_{01}|+\rangle|0\rangle$?

A $|+\rangle|+\rangle$

B $|\Phi^+\rangle$

C $|+\rangle|0\rangle$

D $|00\rangle$

ANSWER

B: $|\Phi^+\rangle$, an entangled state, not a copy

$$\mathrm{CNOT}_{01}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

A copy would be $|+\rangle|+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$$|\langle + +|\Phi^+\rangle|^2 = \left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\cdot 2\right)^2 = \frac{1}{2}$$

Linearity forces a device that copies $|0\rangle$ and $|1\rangle$ to entangle superpositions.

DERIVATION

No-cloning theorem

Claim: no unitary U copies two states with $0 < |\langle\psi|\varphi\rangle| < 1$.

$(\langle\psi\|\langle 0\|)(\|\varphi\rangle\|0\rangle) = \langle\psi\|\varphi\rangle = s$	inputs
$(\langle\psi\|\langle\psi\|)(\|\varphi\rangle\|\varphi\rangle) = s^2$	the copies
$s = s^2$	a unitary preserves inner products
$s = 0 \;\text{ or }\; s = 1$	contradicts 0 < |s| < 1

No device copies an unknown state drawn from a set with two non-orthogonal states.

What can be copied

Not an unknown state from a set with two non-orthogonal members.

A known orthogonal set: $|0\rangle$ and $|1\rangle$ (the CNOT does it); this is how classical information is copied.

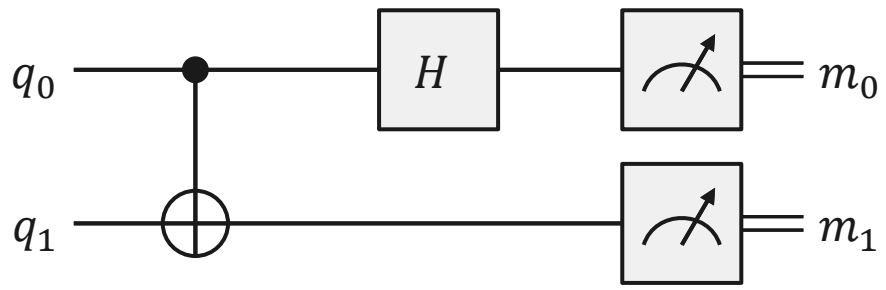
A known description: prepare the state again.

A measurement result: a classical bit, copied freely; the superposition is gone.

5 The Bell measurement

How do we tell the four Bell states apart?

The Bell measurement



CNOT, then H on qubit 0, then measure both

Preparation: $\mathrm{CNOT}_{01}(H \otimes I)$.

Its inverse: $(H \otimes I)\,\mathrm{CNOT}_{01}$: first CNOT, then H.

Each Bell state goes back to its basis state; the meters name it.

Built from local gates and ordinary meters; afterwards the qubits are in a product state.

DERIVATION

Worked example: measuring $|\Psi^+\rangle$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$ input

$$\mathrm{CNOT}_{01}\colon \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = |+\rangle|1\rangle$$ flip q1 when q0 = 1

$$H \otimes I\colon |0\rangle|1\rangle = |01\rangle$$ H|+⟩ = |0⟩

Outcome 01 with certainty: the preparation table read backwards.

YOUR TURN · PREDICT

Your turn: measuring |Φ⁻⟩

Think · 25 seconds

Which outcome $m_0 m_1$ does the Bell measurement give for $|\Phi^-\rangle$?

A 00

B 01

C 10

D 11

ANSWER

C: $|\Phi^-\rangle$ gives 10

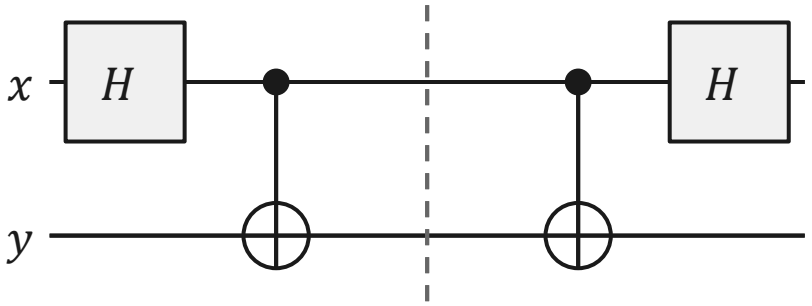
$\mathrm{CNOT}_{01}|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) = |-\rangle|0\rangle$; H gives $|10\rangle$.

$|\Phi^+\rangle \to 00, |\Psi^+\rangle \to 01, |\Phi^-\rangle \to 10, |\Psi^-\rangle \to 11$

SIMULATION

Simulation 4.3: Bell states in Qiskit

```
for xy in ("00", "01", "10", "11"):
    qc.h(0)
    qc.cx(0, 1)
    v = Statevector(qc).reverse_qargs()
    prep = name_of(v.data)
    qc.cx(0, 1)
    qc.h(0)
    p = Statevector(qc).reverse_qargs()
```



prepare | Bell measurement

X gates prepare |xy⟩ (not shown); conversion to textbook order.

Output: input, Bell state, measured bits (from bell_states.out).

```
|00> -> Phi+ -> 00
|01> -> Psi+ -> 01
|10> -> Phi- -> 10
|11> -> Psi- -> 11
```

Recap

1 **Bell states**

H then CNOT: $|xy\rangle \rightarrow$ Φ or Ψ (y), sign (x)

2 **Not just correlation**

Φ^+ agrees in Z and in X; a coin pair only in Z

3 **No-signalling, no-cloning**

Bob's statistics never change; $s = s^2$ rules out copying

4 **Bell measurement**

CNOT, then H, then measure: Φ^+ 00, Ψ^+ 01, Φ^- 10, Ψ^- 11

Next week: Lecture 9

What can a shared Bell pair do? Teleportation, superdense coding, CHSH

Read handout 4.1–4.5; run Simulation 4.2 at home.

Exit ticket

Alice measures her half of $|\Psi^+\rangle$ in the computational basis and obtains 0. What is Bob's qubit now, and what was his probability of 1 before she told him?

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