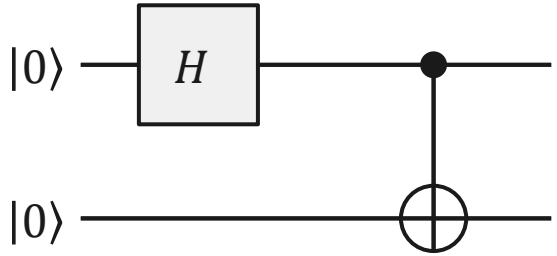
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Two Qubits and Two-Qubit Gates

Tensor products, qubit order and the CNOT



Chapter 3 · Lecture 6

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Today



Two qubits · **Product states** · **Qubit order** · **Operators** · **CNOT** · **More gates**

By the end of this lecture you can

- compute tensor products of states and operators; label basis states and indices
- decide whether a two-qubit state is a product state
- convert Qiskit bit strings and states to textbook order
- write and apply CNOT, CZ, SWAP and controlled-U

YOUR TURN · RECALL

Recap: how big is a three-qubit state?

Recall · 45 seconds

Chapter 1: the state of n qubits is a list of complex amplitudes.

How many amplitudes does three qubits' state vector have?

A 3

B 6

C 8

D 9

ANSWER

C: $2^3 = 8$

Each qubit doubles the list: $2 \times 2 \times 2 = 8$.

Today: why, and what the new joint states can do.

1 Two qubits

How do the states of two systems combine?

The composite-system postulate

Principle · composite systems

$$\text{independent } |a\rangle, |b\rangle \;\Rightarrow\; |a\rangle \otimes |b\rangle$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

Each entry of the first vector multiplies the whole second vector.

n qubits: 2^n amplitudes, the memory wall of Chapter 1.

The last postulate: Chapter 2 had the others.

Labels and indices in textbook order

ket	q0	q1	index	Qiskit prints
$\|00\rangle$	0	0	0	00
$\|01\rangle$	0	1	1	10
$\|10\rangle$	1	0	2	01
$\|11\rangle$	1	1	3	11

$|q_0 q_1\rangle = |q_0\rangle \otimes |q_1\rangle$: qubit 0 leftmost, top wire.

Index $2q_0 + q_1$: qubit 0 is the most significant bit.

Qiskit prints the same state with qubit 0 rightmost (section 3).

Worked example: a tensor product of two states

$$|+\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

1 Qubit 0 in $|+\rangle$, qubit 1 in $|1\rangle$: multiply out.

Before the next slide

Which basis states appear?

Worked example: a tensor product of two states

$$|+\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle), \qquad \text{indices 1 and 3}$$

1 Qubit 0 in $|+\rangle$, qubit 1 in $|1\rangle$: multiply out.

2 Qubit 1 is 1 in both terms; qubit 0 is 0 or 1 with equal amplitudes.

2 Product states

Can every two-qubit state be described qubit by qubit?

Product states and the product test

Definition · product state

$$|\psi\rangle = |a\rangle \otimes |b\rangle$$

Theorem · product test

$$\text{product} \iff a_{00}a_{11} - a_{01}a_{10} = 0$$

Amplitude matrix $M = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$: product ⇔ rank 1 ⇔ det M = 0.

Not a product: entangled (Chapter 4 explains why it matters).

YOUR TURN · CLASSIFY

Your turn: product or not?

Pairs · 90 seconds

A $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

B $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

C $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$

Which are product states? Use the test.

ANSWER

Only A is a product state

A: $\frac{1}{4} - \frac{1}{4} = 0$: it is $|+\rangle \otimes |+\rangle$.

B: $\frac{1}{2} - 0 \neq 0$: entangled.

C: $0 - \frac{1}{3} \neq 0$: entangled, despite equal weights.

Apply the test; equal weights tell you nothing by themselves.

3 Qubit ordering in practice

Which end of a bit string is qubit 0?

Two conventions for one register

Textbook (this course, Cirq, PennyLane, QuTiP)

$|q_0 q_1 \ldots q_{n-1}\rangle$: qubit 0 leftmost, most significant

Qiskit

qubit 0 rightmost in bit strings; least significant in indices

Like big-endian and little-endian byte order.

Neither is wrong; read each result in its own convention.

Test with an asymmetric state: (|00⟩ + |11⟩)/√2 hides the difference.

COMMON MISTAKE

Reading a Qiskit bit string left to right

✗ Qiskit '01' means q0 = 0, q1 = 1

after qc.x(0) Qiskit prints '01'

read left to right: $|01\rangle$

Wrong qubit, wrong answer.

✓ Reverse it first

qubit 0 is the rightmost character

textbook: $|10\rangle$

counts: k[::-1]; states: reverse_qargs()

Every listing converts and says so.

Every multi-qubit Qiskit result in this course is converted to textbook order, with a comment.

SIMULATION

Simulation: one register, two orders

```
for n in (2, 3):
    qc.x(0)
    raw = labels(sv.probabilities_dict())
qc.measure([0, 1], [0, 1])
tb = {k[::-1]: v for k, v in raw.items()}
```

The textbook labels come from reverse_qargs().

Reversing a count key is valid because qubit i is measured into bit i.

circuit	Qiskit	textbook
n = 2: X on q0	'01'	|10⟩
n = 3: X on q0	'001'	|100⟩

```
X on q0, H on q1, 2000 shots:
Qiskit {'01': 955, '11': 1045}
textbook {'10': 955, '11': 1045}
```

4 Operators on several qubits

How does a one-qubit gate act inside a register?

Tensor products of operators

$$A \otimes B = \begin{bmatrix} A_{00}B & A_{01}B \\ A_{10}B & A_{11}B \end{bmatrix}$$

$$(A \otimes B)(|a\rangle \otimes |b\rangle) = A|a\rangle \otimes B|b\rangle$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

X on qubit 1 alone: $I \otimes X$

X on qubit 0 alone: $X \otimes I$

Gates on different qubits can be applied in either order.

WORKED EXAMPLE · STEP 1 OF 2

Worked example: a gate on one qubit of two

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad (I \otimes X)|01\rangle = |00\rangle$$

1 I ⊗ X: two X blocks on the diagonal.

Before the next slide

Now X ⊗ I.

Worked example: a gate on one qubit of two

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad (I \otimes X)|01\rangle = |00\rangle$$

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad (X \otimes I)|01\rangle = |11\rangle$$

1 I ⊗ X: two X blocks on the diagonal.

2 The position of X in the product names the qubit it flips.

YOUR TURN · PREDICT

Your turn: X ⊗ X

Think · 30 seconds

What is $(X \otimes X)|01\rangle$?

A $|00\rangle$

B $|01\rangle$

C $|10\rangle$

D $|11\rangle$

ANSWER

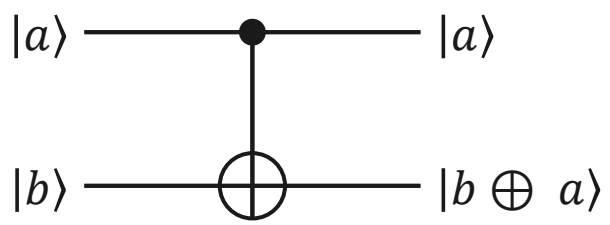
C: both qubits flip

$$(X \otimes X)(|0\rangle \otimes |1\rangle) = X|0\rangle \otimes X|1\rangle = |1\rangle \otimes |0\rangle = |10\rangle$$

5 The CNOT gate

How can one qubit's action depend on another?

CNOT: flip the target if the control is 1



$$\mathrm{CNOT}_{01} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Read the projector form as a sentence: if qubit 0 is $|0\rangle$ do nothing, if $|1\rangle$ apply X.

Swaps $|10\rangle$ and $|11\rangle$.

On basis states: the reversible XOR of digital logic.

YOUR TURN · PREDICT

Predict: CNOT on a superposed control

Predict · 1 minute

Apply CNOT_{01} to $|+\rangle|0\rangle$. What comes out?

A $|+\rangle|0\rangle$: the control is not 1

B $|+\rangle|+\rangle$: the control is copied

C $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

D $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

ANSWER

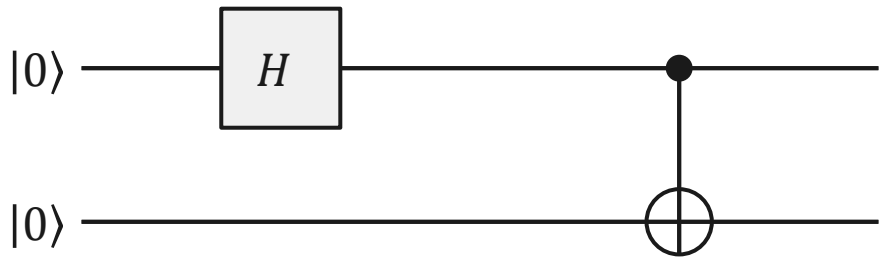
D: the gate acts on each branch

$$\mathrm{CNOT}_{01}\,\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Product test: $a_{00}a_{11} - a_{01}a_{10} = \frac{1}{2} \neq 0$: entangled.

A product input became an entangled output: no tensor product A ⊗ B can do that.

The same steps as a circuit



start	$\|00\rangle$	product
after H	$\frac{1}{\sqrt{2}}(\|00\rangle + \|10\rangle)$	product
after CNOT	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	not a product

Each time slice is a matrix-vector product.

The entanglement appears at the CNOT.

Chapter 4 starts here: this is a Bell state.

COMMON MISTAKE

“A superposed control does nothing, or copies itself”

✗ Control must be 0 or 1

no definite control, so the gate is idle

or: the target becomes a copy, $|+\rangle|+\rangle$

Both predictions are wrong.

✓ The gate acts branch by branch

$|00\rangle \to |00\rangle, |10\rangle \to |11\rangle$

one entangled state of both qubits

no circuit copies an unknown state (Ch. 4)

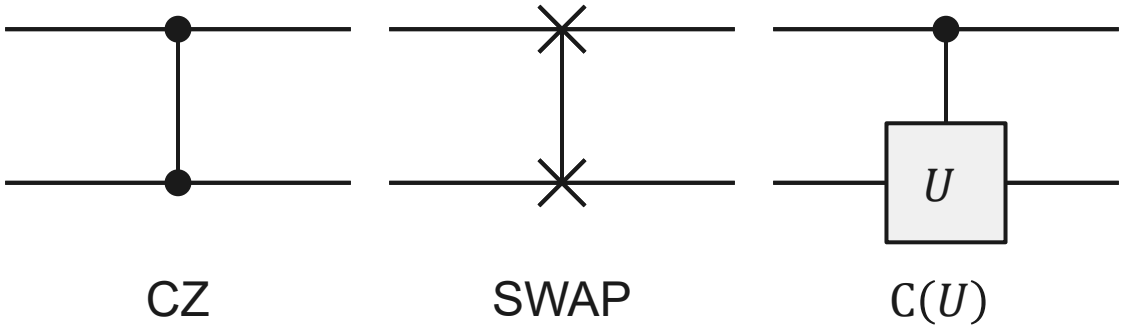
Linearity decides, not intuition.

A controlled gate acts on every branch of a superposed control; it neither idles nor clones.

6 CZ, SWAP and controlled-U

Which other two-qubit gates appear constantly?

Three more two-qubit gates



CZ is symmetric: either qubit is the control.

SWAP: $|a\rangle|b\rangle \to |b\rangle|a\rangle$, three CNOTs.

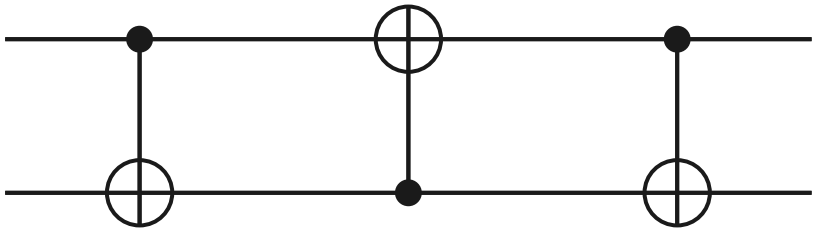
CNOT = C(X); CZ = C(Z).

$$\mathrm{CZ} = \mathrm{diag}(1,1,1,-1)$$

$$\mathrm{C}(U) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$$

WORKED EXAMPLE · STEP 1 OF 2

Worked example: SWAP from three CNOTs



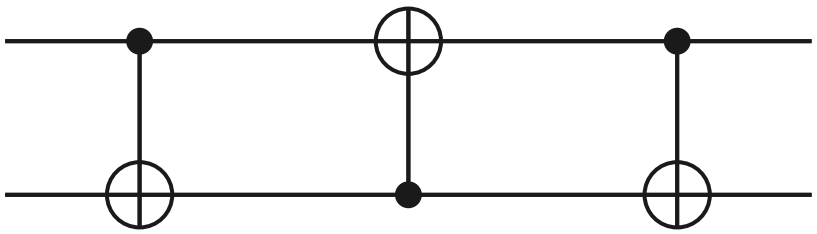
input	CNOT01
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

1 Follow each basis state through the first CNOT.

Before the next slide

Continue through the other two.

Worked example: SWAP from three CNOTs



input	CNOT01	CNOT10	CNOT01
$\|00\rangle$	$\|00\rangle$	$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$	$\|11\rangle$	$\|10\rangle$
$\|10\rangle$	$\|11\rangle$	$\|01\rangle$	$\|01\rangle$
$\|11\rangle$	$\|10\rangle$	$\|10\rangle$	$\|11\rangle$

1 Follow each basis state through the first CNOT.

2 Every $|a\rangle|b\rangle$ ends as $|b\rangle|a\rangle$: the three-XOR exchange of two registers.

A controlled global phase is a relative phase

$$\mathrm{C}\left(e^{i\gamma} I\right) = |0\rangle\langle 0| \otimes I + e^{i\gamma}|1\rangle\langle 1| \otimes I$$

$$= \mathrm{P}(\gamma) \otimes I$$

Alone, a global phase is invisible (Chapter 2).

Controlled, it multiplies only the |1⟩ branch of the control: a relative phase.

So C(Rz(θ)) and C(P(θ)) are different gates.

SIMULATION

Simulation: the gate matrices, checked

```
CX01 = np.kron(P0, I) + np.kron(P1, X)
CX10 = np.kron(I, P0) + np.kron(X, P1)
CZ = np.kron(P0, I) + np.kron(P1, Z)
SWAP = CX01 @ CX10 @ CX01
    return Operator(qc).reverse_qargs().data
```

NumPy builds each gate from projectors; Qiskit's gates are converted to textbook order.

Last line: Qiskit's raw cx matrix is in Qiskit's own order.

check	result
CNOT = qiskit cx(0, 1)	True
CZ = qiskit cz(0, 1)	True
CZ symmetric: cz(1, 0)	True
SWAP = qiskit swap(0, 1)	True
raw Qiskit cx = CNOT?	**False**

Recap

1

Two qubits

$|a\rangle \otimes |b\rangle$; 2^n amplitudes; qubit 0 leftmost

2

Product test

$a_{00}a_{11} - a_{01}a_{10} = 0$; otherwise entangled

3

Qiskit order

qubit 0 rightmost: reverse keys, reverse_qargs()

4

Controlled gates

CNOT, CZ, SWAP, $\mathrm{C}(U)$; entangle product inputs

Next: Lecture 7

The circuit model: measurement, universality, transpilation

Read: handout 3.1–3.5; then 3.6–3.10 before Lecture 7.

Exit ticket

Is $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$ a product state? If so, find the two single-qubit states.

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