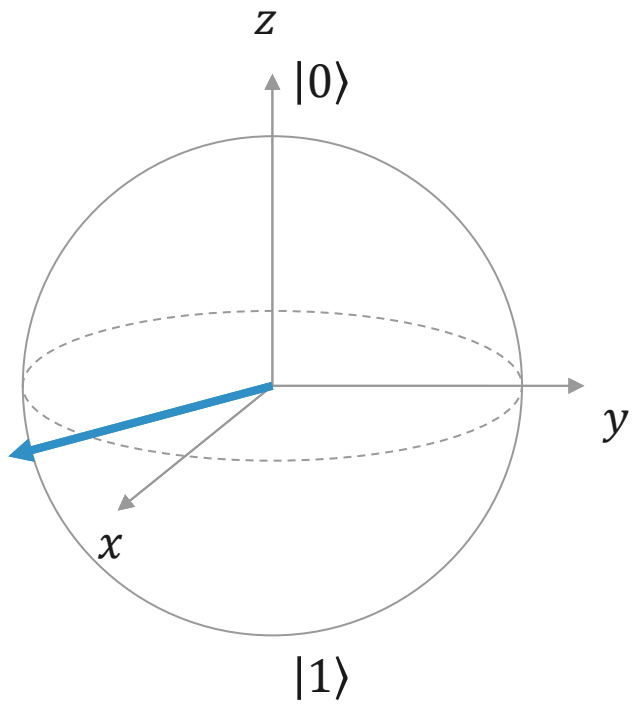
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Dynamics and Observables

How qubits evolve, and what a measurement averages

Chapter 2 · Lecture 5



Sajid Muhaimin Choudhury

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Today

1. **Schrödinger equation**
2. **Precession**
3. **Rabi oscillation**
4. **Observables**
5. **Shots**

By the end of this lecture you can

- solve the Schrödinger equation for a constant 2×2 Hamiltonian
- describe Larmor precession and resonant Rabi oscillation; relate π and π/2 pulses to gates
- compute expectation values and variances; measure X or Y with a basis change
- judge simulated estimates with the standard error $1/\sqrt{N}$

YOUR TURN · RECALL

Recap: Rz(π/2) and S

Recall · 1 minute

Write $\mathrm{R}_z\left(\frac{\pi}{2}\right)$ as a matrix. Which relation holds?

A $S = \mathrm{R}_z\left(\frac{\pi}{2}\right)$

B $S = e^{i\pi/4}\mathrm{R}_z\left(\frac{\pi}{2}\right)$

C $S = e^{i\pi/2}\mathrm{R}_z\left(\frac{\pi}{2}\right)$

D they differ by a relative phase

ANSWER

B: equal up to a global phase

$$\mathrm{R}_z\left(\frac{\pi}{2}\right) = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = e^{-i\pi/4}S$$

Today: where do such rotations come from physically?

1 The Schrödinger equation

How does a qubit change in time when nobody measures it?

The Schrödinger equation

Principle · evolution (continuous form)

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{\mathrm{H}} |\psi(t)\rangle$$

Constant Hamiltonian

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \qquad U(t) = e^{-i\hat{\mathrm{H}}t/\hbar}$$

$\hat{\mathrm{H}}$: the Hamiltonian, Hermitian; eigenvalues are energies.

H stays the Hadamard gate.

Spectral form: $U(t) = \sum_k e^{-iE_k t/\hbar} |E_k\rangle\langle E_k|$, unitary at every instant.

Code sets ħ = 1.

A state equation that conserves probability

$$\dot{\mathrm{x}} = Ax \;\Rightarrow\; x(t) = e^{At}x(0)$$

$$A = -i\hat{\mathrm{H}}/\hbar\text{:}\quad \text{eigenvalues}\; -iE_k/\hbar$$

$$|E_k\rangle \;\rightarrow\; e^{-iE_k t/\hbar}|E_k\rangle$$

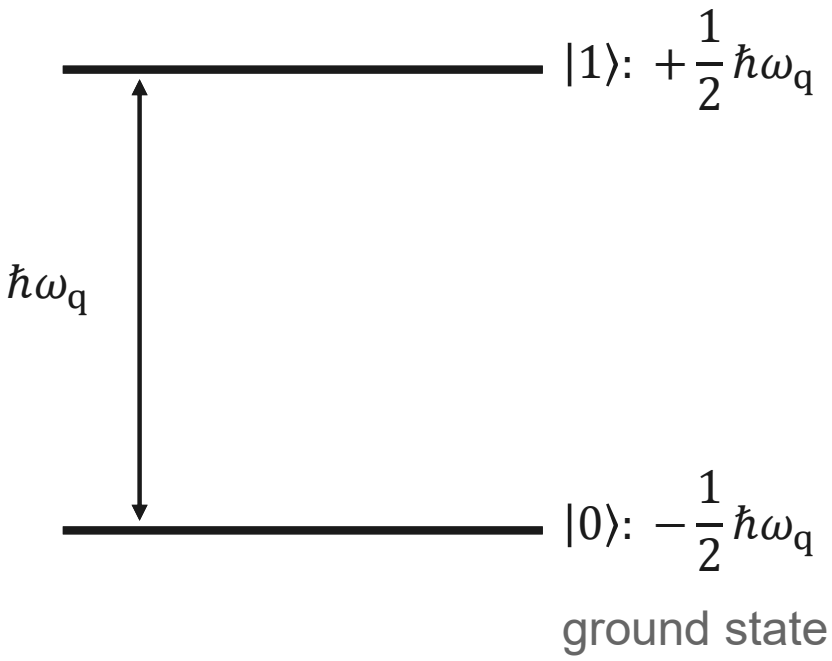
stationary: only a global phase

Purely imaginary eigenvalues: every mode oscillates, none grows or decays.

So the norm stays 1.

Breaks down: real qubits lose energy (Chapter 7).

The two-level Hamiltonian



$$\hat{H}_0 = -\frac{\hbar\omega_q}{2}Z$$

Ground state $|0\rangle$ at the north pole (Krantz et al. 2019).

$f_q = \omega_q/2\pi$: the transition frequency.

Many texts write +½ħωZ; signs differ in the literature.

YOUR TURN · PREDICT

Predict: $|+\rangle$ under $\hat{H}_0$

Predict · 1 minute

A qubit starts in $|+\rangle$ and evolves under $\hat{H}_0$. Where is it after a quarter period $t = \frac{\pi}{2\omega_q}$?

A $|+i\rangle$

B $|-i\rangle$

C $|-\rangle$

D $|1\rangle$

ANSWER

B: it precesses towards |−i⟩

Diagonal $\hat{\mathrm{H}}_0$: only the relative phase of $|1\rangle$ changes, $\varphi(t) = -\omega_{\mathrm{q}} t$.

After a quarter period $\varphi = -\frac{\pi}{2}$: the state $|-i\rangle$.

Z-basis probabilities stay ½, ½ throughout.

WORKED EXAMPLE · STEP 1 OF 2

Worked example: Larmor precession

$$U(t) = e^{i\omega_q tZ/2} = \mathrm{diag}\left(e^{i\omega_q t/2}, e^{-i\omega_q t/2}\right)$$

$$U(t)|+\rangle = e^{i\omega_q t/2}\frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega_q t}|1\rangle\right)$$

1 Diagonal Hamiltonian: exponentiate the diagonal.

Before the next slide

Factor out a global phase.

Worked example: Larmor precession

$$U(t) = e^{i\omega_q t Z/2} = \mathrm{diag}\left(e^{i\omega_q t/2}, e^{-i\omega_q t/2}\right)$$

$$U(t)|+\rangle = e^{i\omega_q t/2}\frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega_q t}|1\rangle\right)$$

$$\mathbf{r}(t) = \left(\cos\omega_q t, -\sin\omega_q t, 0\right)$$

1 Diagonal Hamiltonian: exponentiate the diagonal.

2 Azimuth $\varphi(t) = -\omega_q t$, $\theta = \pi/2$: the equator, clockwise seen from +z.

SIMULATION

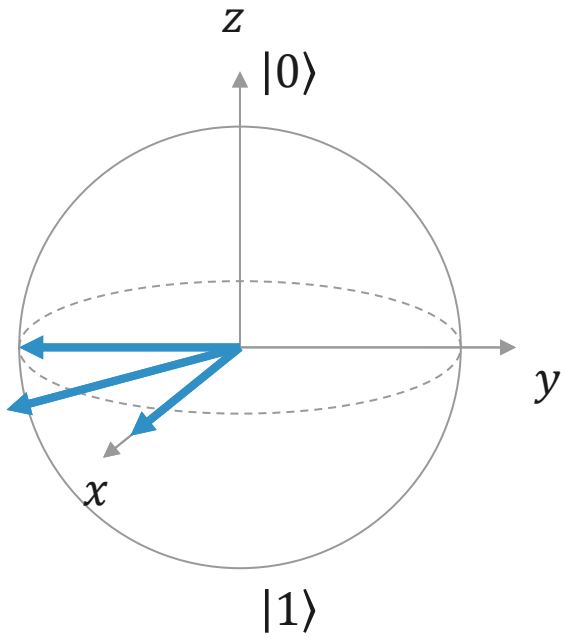
Simulation: Larmor precession in QuTiP

```
wq = 2 * np.pi
H = -(wq / 2) * qt.sigmaz()
times = [0, 0.125, 0.25, 0.5, 0.75, 1.0]
res = qt.sesolve(H, plus, times,
```

$\hbar = 1, \omega_q = 2\pi$: one period = one time unit.

Starts in $|+\rangle$; sesolve returns $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$.

Sphere: t = 0, 1/8, 1/4 (from +x towards −y).



t	<X>	<Y>
0.000	+1.000	+0.000
0.125	+0.707	-0.707
0.250	+0.000	-1.000
0.500	-1.000	+0.000

2 Driving the qubit

How does a pulse turn $|0\rangle$ into $|1\rangle$?

EE LINK

Resonant drive and the rotating frame

Resonant drive, rotating frame (stated)

$$\hat{\mathrm{H}}_{\mathrm{R}} = \frac{\hbar\Omega}{2} X$$

Drive at $\omega_{\mathrm{d}} \approx \omega_{\mathrm{q}}$ (microwave or laser).

Ω: Rabi frequency $\propto$ drive amplitude.

A receiver mixes a carrier down to baseband.

Rotating frame: precession at ω_{q} is the carrier; Rabi oscillation at Ω is the envelope.

Dropping the $2\omega_{\mathrm{d}}$ term = rotating-wave approximation.

Worked example: Rabi oscillation

$$U(t) = e^{-i\Omega\, tX/2} = \mathrm{R}_x(\Omega\, t)$$

1 Exponentiate the constant Hamiltonian: a rotation about x.

Before the next slide

Apply it to |0⟩.

Worked example: Rabi oscillation

$$U(t) = e^{-i\Omega\, tX/2} = \mathrm{R}_x(\Omega\, t)$$

$$U(t)|0\rangle = \cos\frac{\Omega\, t}{2}|0\rangle - i\sin\frac{\Omega\, t}{2}|1\rangle \;\Rightarrow\; \Pr(1) = \sin^2\frac{\Omega\, t}{2}$$

1 Exponentiate the constant Hamiltonian: a rotation about x.

2 First column of the rotation gives the amplitudes.

Before the next slide

Which duration gives X?

Worked example: Rabi oscillation

$$U(t) = e^{-i\Omega\, tX/2} = \mathrm{R}_x(\Omega\, t)$$

$$U(t)|0\rangle = \cos\frac{\Omega\, t}{2}|0\rangle - i\sin\frac{\Omega\, t}{2}|1\rangle \;\Rightarrow\; \Pr(1) = \sin^2\frac{\Omega\, t}{2}$$

$$t = \frac{\pi}{\Omega}\colon\ \mathrm{R}_x(\pi) = -iX; \qquad t = \frac{\pi}{2\Omega}\colon\ \mathrm{R}_x\left(\frac{\pi}{2}\right)$$

1 Exponentiate the constant Hamiltonian: a rotation about x.

2 First column of the rotation gives the amplitudes.

3 π pulse: NOT up to a global phase. π/2 pulse: equal superposition.

SIMULATION

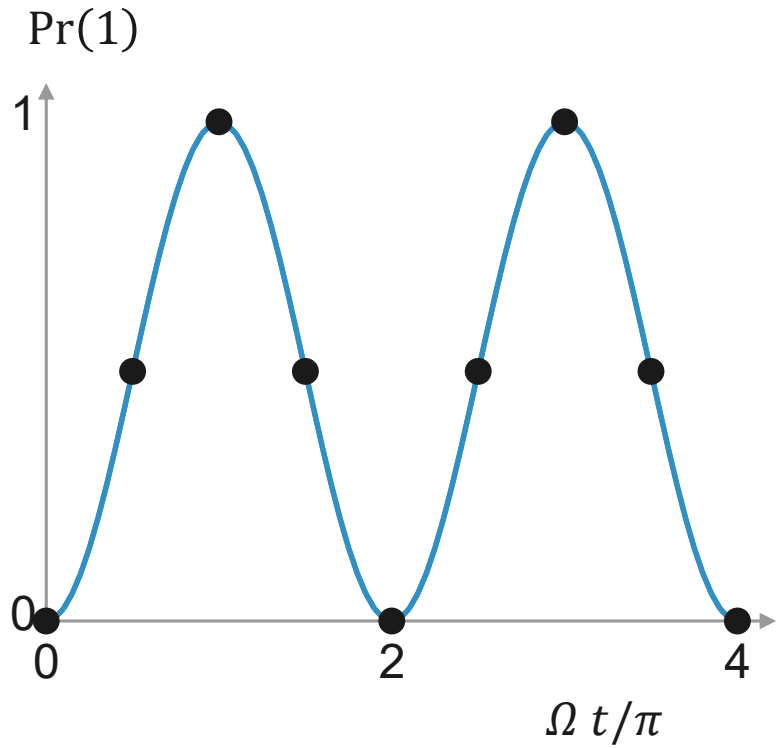
Simulation: Rabi oscillation in QuTiP

```
H_res = (W / 2) * qt.sigmax()
P1 = qt.basis(2, 1).proj()
def p1(H, times):
    res = qt.sesolve(H, qt.basis(2, 0), times,
grid = np.pi / 2 * np.arange(9)
```

$\Omega = 1$, start in $|0\rangle$.

π/2 pulse (t = 1.571): P1 = 0.500; π pulse (t = 3.142): P1 = 1.000.

Curve: sin²(Ωt/2); dots: simulated.



3 Observables

What does a measurement of a physical quantity return?

Observables are Hermitian operators

Principle · observables

$$A = A^\dagger = \sum_k a_k |a_k\rangle\langle a_k|$$

- result: an eigenvalue a_k
- probability $|\langle a_k|\psi\rangle|^2$
- afterwards $|a_k\rangle$

Measuring A = measuring in its eigenbasis, eigenvalues as labels.

Z: +1 for $|0\rangle$, −1 for $|1\rangle$.

Only Hermitian operators: S is a gate, not an observable.

Expectation value and variance

Definition · expectation value

$$\langle A\rangle = \sum_k a_k |\langle a_k|\psi\rangle|^2 = \langle\psi|A|\psi\rangle$$

Variance

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$$

The mean of many results on identically prepared copies.

No eigenvectors needed: $\langle\psi|A|\psi\rangle$

Pauli $P^2 = I$: $(\Delta P)^2 = 1 - \langle P\rangle^2$

COMMON MISTAKE

“Applying Z is measuring Z”

✗ The matrix gives the result

$Z|+\rangle = |-\rangle$

so the result is $|-\rangle$?

A gate is deterministic and reversible.

✓ A measurement is random

Measuring Z on $|+\rangle$: ±1, ½ each

leaves $|0\rangle$ or $|1\rangle$

$\langle +|Z|+\rangle = 0$ summarises the statistics

Only $\langle \psi|Z|\psi\rangle$ summarises the statistics.

The same matrix plays two unrelated roles: a gate, and an observable whose eigenbasis is measured.

The Bloch vector is measurable

$$\langle Z\rangle = |\alpha|^2 - |\beta|^2 = \cos\theta$$

$$\langle X\rangle = 2\,\mathrm{Re}(\alpha^*\beta) = \sin\theta\cos\varphi$$

$$\langle Y\rangle = 2\,\mathrm{Im}(\alpha^*\beta) = \sin\theta\sin\varphi$$

$$\mathbf{r} = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$$

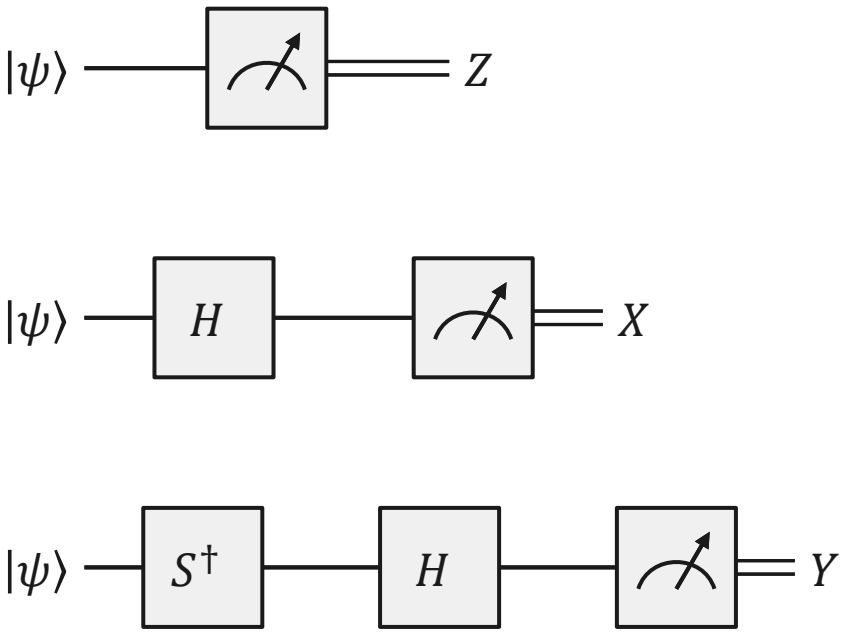
With $\alpha = \cos\frac{\theta}{2}, \beta = e^{i\varphi}\sin\frac{\theta}{2}$

and $2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta$.

Each component is the average result of one Pauli measurement.

Simulation 2.3 (Lecture 4) computed Bloch vectors this way.

Measuring X and Y with a basis change



X: H maps $|\pm\rangle$ to $|0\rangle, |1\rangle$

Y: $S^\dagger$ maps $|\pm i\rangle$ to $|\pm\rangle$, then H

Outcome 0 means +1, outcome 1 means −1.

YOUR TURN · CALCULATE

Your turn: expectation values

Pairs · 2 minutes

For $|\psi\rangle = \mathrm{R}_y\left(\frac{\pi}{3}\right)|0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, find $\langle Z\rangle, \langle X\rangle$ and the variances of Z and X.

Which observable has the smaller spread?

ANSWER

$\langle Z \rangle = 0.5$, $\langle X \rangle \approx 0.866$: X spreads less

$$\langle Z \rangle = \frac{3}{4} - \frac{1}{4} = 0.5, \qquad \langle X \rangle = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \approx 0.866$$

$$(\Delta Z)^2 = 1 - 0.25 = 0.75, \qquad (\Delta X)^2 = 1 - 0.75 = 0.25$$

Bloch vector $(0.866, 0, 0.5)$ (Lecture 4).

4 Estimating from shots

How precise is an average of N results?

The standard error falls as 1/√N

Estimate of a Pauli expectation value

$$\bar{A} = \frac{n_0 - n_1}{N}, \qquad \mathrm{SE} = \frac{\Delta A}{\sqrt{N}} = \sqrt{\frac{1 - \langle A \rangle^2}{N}}$$

One more decimal digit costs 100 times more shots.

Eigenstate ($\langle A \rangle = \pm 1$): one shot suffices.

$\langle A \rangle = 0$: the most expensive case.

YOUR TURN · PREDICT

Predict, then simulate: the standard error

Predict · 1 minute

Estimate $\langle Z \rangle$ of $\mathrm{R}_y\left(\frac{\pi}{3}\right)|0\rangle$ from $N = 1000$ shots. Standard error?

A 0.0005

B 0.0158

C 0.0274

D 0.0866

ANSWER

C: $\sqrt{(0.75/1000)} \approx 0.0274$

$$\Delta Z = \sqrt{0.75} \approx 0.866, \qquad \mathrm{SE} = \frac{0.866}{\sqrt{1000}} \approx 0.0274$$

The simulator prints the same standard error; the estimate itself scatters by about that much.

SIMULATION

Simulation: expectation values from shots

```
sim = AerSimulator(seed_simulator=6523)
def estimate(basis, shots):
    qc.ry(np.pi / 3, 0)
    if basis == "X":
        qc.h(0)
se = np.sqrt((1 - exact[b] ** 2) / n)
```

N	<Z>	<X>
100	0.480 ± 0.0866	0.880 ± 0.0500
1000	0.526 ± 0.0274	0.880 ± 0.0158
10000	0.498 ± 0.0087	0.864 ± 0.0050

exact: $\langle Z \rangle = 0.5$, $\langle X \rangle \approx 0.866$

X is measured with H before the meter.

Every estimate lies within about one standard error; SE shrinks by $\sqrt{10}$ per decade.

Recap

1

Dynamics

$U(t) = e^{-i\hat{\mathrm{H}}t/\hbar}$; energy eigenstates only gain a phase

2

Precession and Rabi

precession about z at ω_{q}; $\sin^2(\Omega\, t/2)$

3

Observables

Hermitian; $\langle A\rangle = \langle\psi|A|\psi\rangle$

4

Shots

standard error $\sqrt{(1-\langle A\rangle^2)/N}$

Next: Lecture 6 (Chapter 3)

Composite systems: two qubits and two-qubit gates

Read: handout 2.8–2.9; self-study 2.10–2.11 (detuning, Ramsey, uncertainty).

Exit ticket

How many shots are needed so that the standard error of $\langle Z \rangle$ for $\mathrm{R}_y\left(\frac{2\pi}{3}\right)|0\rangle$ is at most 0.01?

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