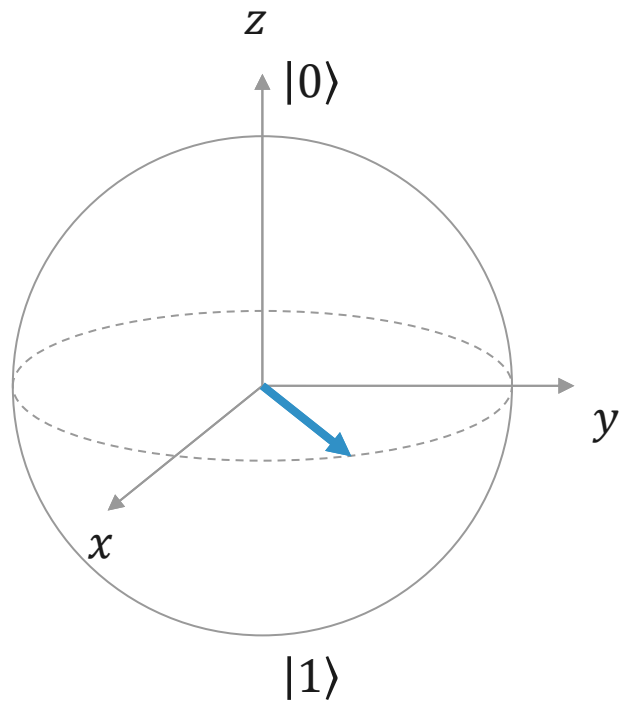
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Single-Qubit Gates and the Bloch Sphere

Every gate is a rotation

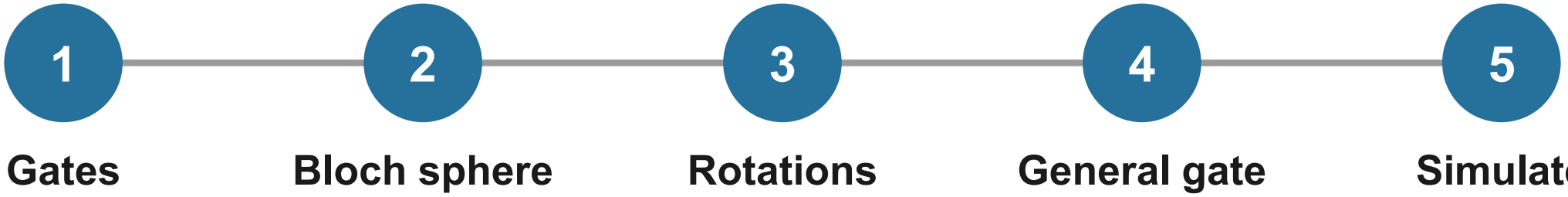


Chapter 2 · Lecture 4

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Today



By the end of this lecture you can

- write the matrices of X, Y, Z, H, S, T and P(φ) and use their identities
- convert a state to Bloch angles and a Bloch vector; orthogonal states are antipodal
- treat gates as rotations $\mathrm{R}_z(\theta) = e^{-i\theta\, Z/2}$ and track global phases
- choose rotation angles for a target probability; read Qiskit's gate names

YOUR TURN · RECALL

Recap: the exit ticket

Recall · 1 minute

Can $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ be distinguished …

… by a Z measurement? By some measurement?

A no; no

B no by Z; yes by another basis

C yes by Z; yes

D yes by Z; no other basis

1 The single-qubit gates

Which fixed matrices does every circuit use?

Pauli gates and the Hadamard

Definition · Pauli and Hadamard gates

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

X: bit flip. Z: phase flip, $Z|+\rangle = |-\rangle$

$Y = iXZ$: both, with a phase.

H: $|0\rangle \leftrightarrow |+\rangle, |1\rangle \leftrightarrow |-\rangle$

All four are Hermitian and unitary: $X^2 = Y^2 = Z^2 = H^2 = I$

Phase gates: P(φ), S and T

Definition · phase gate

$$\mathrm{P}(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$

Special cases

$$S = \mathrm{P}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$T = \mathrm{P}\left(\frac{\pi}{4}\right), \qquad Z = \mathrm{P}(\pi)$$

Phases add: $\mathrm{P}(\varphi_1)\mathrm{P}(\varphi_2) = \mathrm{P}(\varphi_1 + \varphi_2)$

Hence $T^2 = S, S^2 = Z$.

Not Hermitian: $S^\dagger = \mathrm{P}\left(-\frac{\pi}{2}\right) \neq S$

Identities: $HXH = Z, \quad HZH = X$

YOUR TURN · CALCULATE

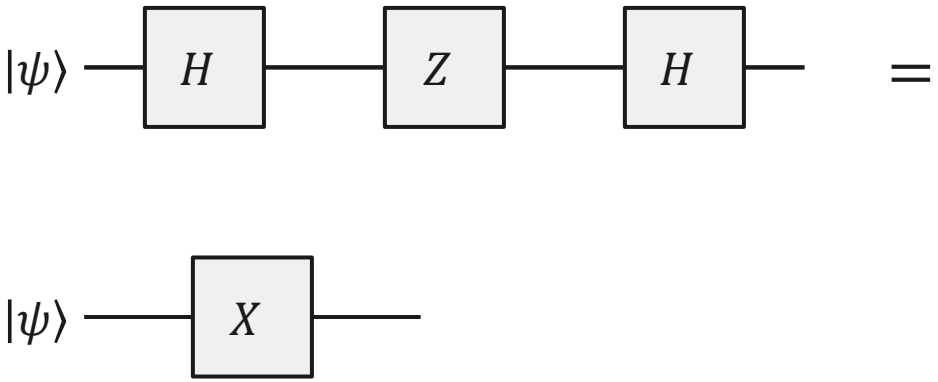
Your turn: compute HZH

Pairs · 2 minutes

Multiply the matrices to find HZH. Then find HYH.

Read HZH as a circuit: what does it do?

Reading an identity as a circuit



change basis · act · change back

A circuit lists gates in time order.

The matrix product is written in reverse: $H\,Z\,H$ happens to be symmetric.

Basis change, act, change back: a pattern that returns in Chapter 5.

2 The Bloch sphere

How many real numbers does a qubit state really need?

Two real parameters

$$4 \text{ real numbers } - \ 1 \text{ (norm) } - \ 1 \text{ (global phase) } = 2$$

Every pure state, up to a global phase

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

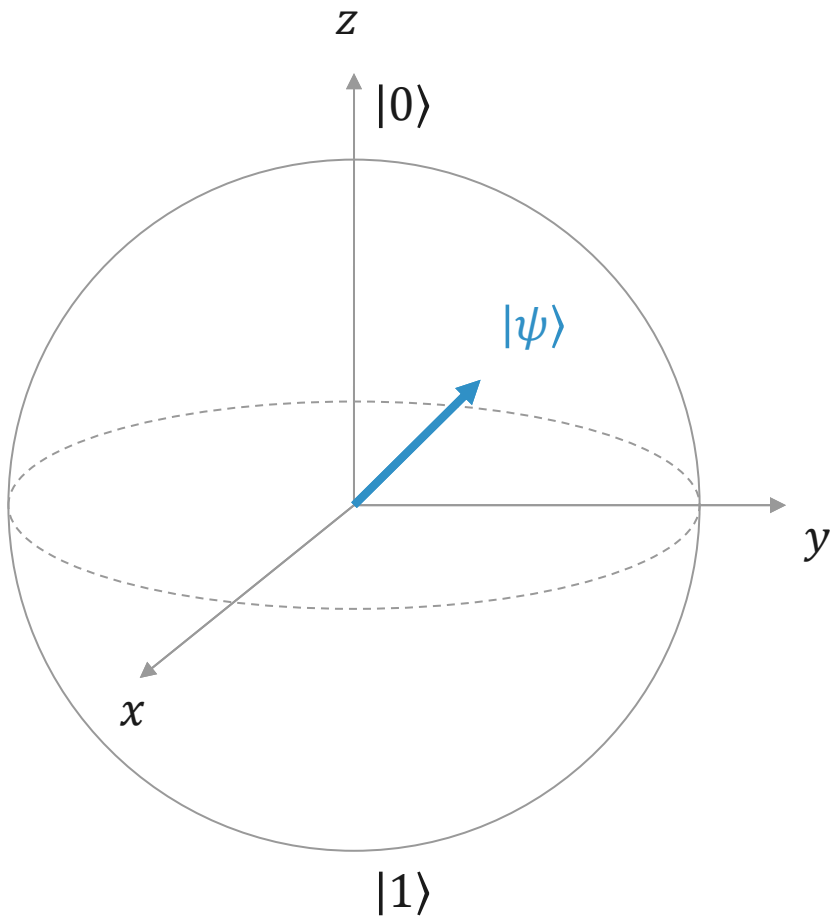
$$0 \le \theta \le \pi, \qquad 0 \le \varphi < 2\pi$$

Write $\alpha = |\alpha|e^{ia}, \beta = |\beta|e^{ib}$.

Drop e^{ia}; set $\varphi = b - a$.

$|\alpha| = \cos\frac{\theta}{2}$ with $\frac{\theta}{2} \in [0, \frac{\pi}{2}]$.

Bloch angles and the Bloch vector

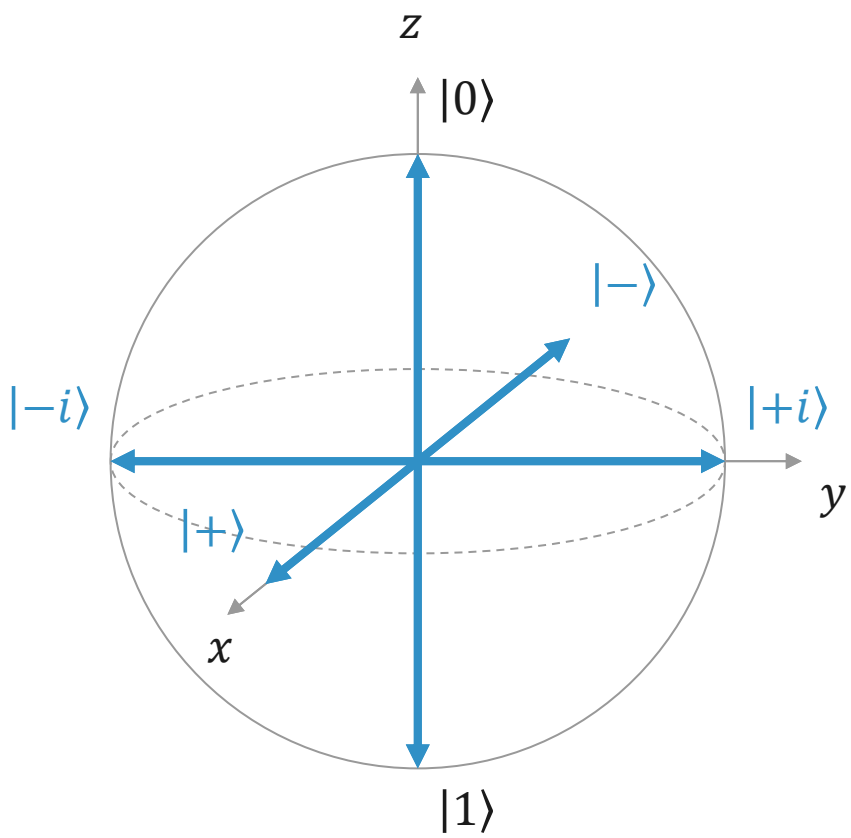


$\mathbf{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

θ from +z; φ from +x in the equatorial plane.

Lecture 5: $\mathbf{r} = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$, measurable averages.

Six cardinal states



z: $|0\rangle, |1\rangle$

x: $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

y: $|\pm\, i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm\ i|1\rangle)$

Each axis carries one orthonormal basis.

Orthogonal means antipodal

Theorem · overlap and Bloch vectors

$$|\langle\varphi|\psi\rangle|^2 = \frac{1 + \mathbf{r}_\varphi \cdot \mathbf{r}_\psi}{2}$$

Corollary

$$\langle\varphi|\psi\rangle = 0 \;\Leftrightarrow\; \mathbf{r}_\varphi = -\mathbf{r}_\psi$$

Angles double: 90° in Hilbert space, 180° on the sphere.

Check with $|\varphi\rangle = |0\rangle$: $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

Other states: rotate to $|0\rangle$ (next section).

WORKED EXAMPLE · STEP 1 OF 3

Worked example: finding Bloch angles

$$|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle): \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}$$

1 Match the modulus of the |0⟩ amplitude to cos(θ/2).

Before the next slide

What is φ for the amplitude i?

Worked example: finding Bloch angles

$$|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle): \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}$$

$$e^{i\varphi} = i \Rightarrow \varphi = \frac{\pi}{2}, \qquad \mathbf{r} = (0,1,0) = |+i\rangle$$

1 Match the modulus of the |0⟩ amplitude to cos(θ/2).

2 The phase of the |1⟩ amplitude is φ.

Before the next slide

Now |b⟩ = (√3/2)|0⟩ + (1/2)|1⟩.

WORKED EXAMPLE · STEP 3 OF 3

Worked example: finding Bloch angles

$$|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle): \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}$$

$$e^{i\varphi} = i \Rightarrow \varphi = \frac{\pi}{2}, \qquad \mathbf{r} = (0,1,0) = |+i\rangle$$

$$|b\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle: \quad \theta = \frac{\pi}{3}, \varphi = 0, \mathbf{r} \approx (0.866,0,0.5)$$

1 Match the modulus of the |0⟩ amplitude to cos(θ/2).

2 The phase of the |1⟩ amplitude is φ.

3 The θ/2 trap: θ is π/2 and π/3, not π/4 and π/6.

3 Rotations

What does a gate do to the sphere?

Rotation gates

Definition · rotation gates (c = cos θ/2, s = sin θ/2)

$$\mathrm{R}_x(\theta) = e^{-i\theta\, X/2} = \begin{bmatrix} c & -is \\ -is & c \end{bmatrix}$$

$$\mathrm{R}_y(\theta) = e^{-i\theta\, Y/2} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\mathrm{R}_z(\theta) = e^{-i\theta\, Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

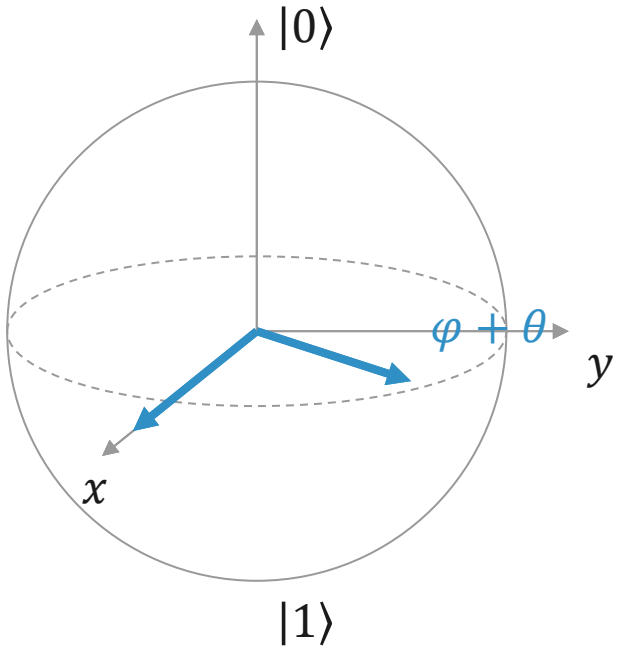
Chapter 1: $e^{-i\theta\, X/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X$

The same construction with Y and Z.

$\mathrm{R}_y(\theta)$ is a real rotation matrix.

A turn about z changes the azimuth

$$\mathrm{R}_z(\theta)|\psi\rangle = e^{-i\theta/2}\left(\cos\frac{\theta_0}{2}|0\rangle + e^{i(\varphi+\theta)}\sin\frac{\theta_0}{2}|1\rangle\right)$$



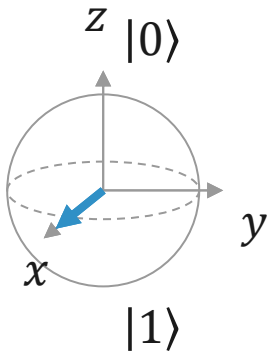
Up to a global phase, φ → φ + θ.

A turn through θ about z, anticlockwise seen from +z.

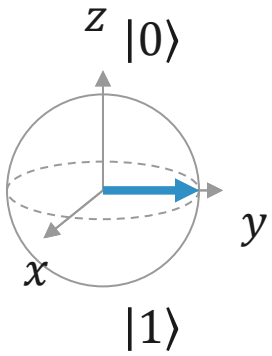
R_x and R_y: turns through θ about x and y.

$\mathrm{R}_y(\theta)|0\rangle$ has Bloch angles $(\theta, 0)$.

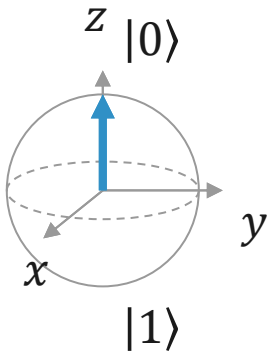
Every fixed gate is a rotation



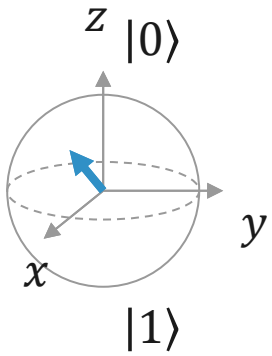
X: π about x



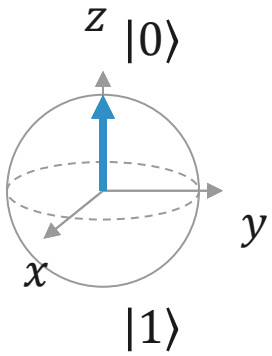
Y: π about y



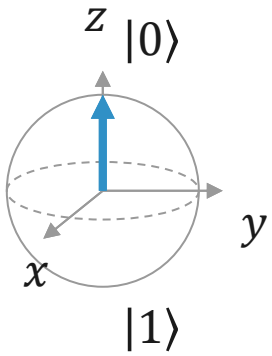
Z: π about z



H: π about $(x + z)$



S: $\pi/2$ about z



T: $\pi/4$ about z

$X = i\mathrm{R}_x(\pi)$, likewise Y, Z

$$S = e^{i\pi/4}\mathrm{R}_z\left(\frac{\pi}{2}\right)$$

$$T = e^{i\pi/8}\mathrm{R}_z\left(\frac{\pi}{4}\right)$$

$$\mathrm{P}(\varphi) = e^{i\varphi/2}\mathrm{R}_z(\varphi)$$

COMMON MISTAKE

“Rz(π) is just Z”

✗ Same gate, same matrix

$\mathrm{R}_z(\pi) = Z$

so they can be swapped anywhere

np.allclose says False.

✓ Equal up to a global phase

$$\mathrm{R}_z(\pi) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -iZ$$

identical on one qubit

different inside a controlled gate (Ch. 3)

Software follows the literal definition.

Alone, $\mathrm{R}_z(\pi)$ acts like Z; under a control the global phase becomes a relative phase.

YOUR TURN · CALCULATE

Your turn: prepare P(1) = 0.25

Think · 1 minute, then vote

Choose θ so that $\mathrm{R}_y(\theta)|0\rangle$ gives outcome 1 with probability 0.25.

A $\frac{\pi}{6}$

B $\frac{\pi}{4}$

C $\frac{\pi}{3}$

D $\frac{\pi}{2}$

ANSWER

C: θ = π/3

$$\Pr(1) = \sin^2\frac{\theta}{2} = 0.25 \;\Rightarrow\; \sin\frac{\theta}{2} = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$$

Bloch vector $(0.866, 0, 0.5)$, $\Pr(0) = \frac{1+z}{2} = 0.75$

In general $\theta = 2\arcsin\sqrt{p}$; Qiskit: qc.ry(np.pi/3, 0)

4 The general single-qubit gate

How many rotations does any gate need?

Any gate from three rotations

Theorem · Z–Y decomposition (Nielsen & Chuang, Thm 4.1)

$$U = e^{i\alpha}\,\mathrm{R}_z(\beta)\,\mathrm{R}_y(\gamma)\,\mathrm{R}_z(\delta)$$

Qiskit's general gate u(θ, φ, λ)

$$\begin{bmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & e^{i(\varphi+\lambda)}\cos\frac{\theta}{2} \end{bmatrix}$$

$$= e^{i(\varphi+\lambda)/2}\,\mathrm{R}_z(\varphi)\,\mathrm{R}_y(\theta)\,\mathrm{R}_z(\lambda)$$

Rotations about two fixed axes suffice.

Stated here; proof in the reference section.

One parametrised gate stands for every single-qubit operation.

Qiskit's names for these gates

rx, ry, rz	$\mathrm{R}_x, \mathrm{R}_y, \mathrm{R}_z$ exactly
p(φ)	$\mathrm{P}(\varphi) = \mathrm{diag}\left(1, e^{i\varphi}\right)$
s, t, sdg	$S, T, S^\dagger$
u(θ, φ, λ)	$e^{i(\varphi+\lambda)/2}\,\mathrm{R}_z(\varphi)\mathrm{R}_y(\theta)\mathrm{R}_z(\lambda)$

Other books: T is the 'π/8 gate', and Rz sometimes means P (self-study 2.11).

5 Simulating gates and the sphere

Does the simulator agree with the geometry?

YOUR TURN · PREDICT

Predict, then simulate: where is TH|0⟩?

Predict · 1 minute

Apply H, then T, to $|0\rangle$. Where is the state on the Bloch sphere?

A $(\theta, \varphi) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$: equator, between +x and +y

B $\left(\frac{\pi}{4}, 0\right)$

C $\left(\frac{\pi}{2}, 0\right)$: T changes nothing visible

D the north pole

SIMULATION

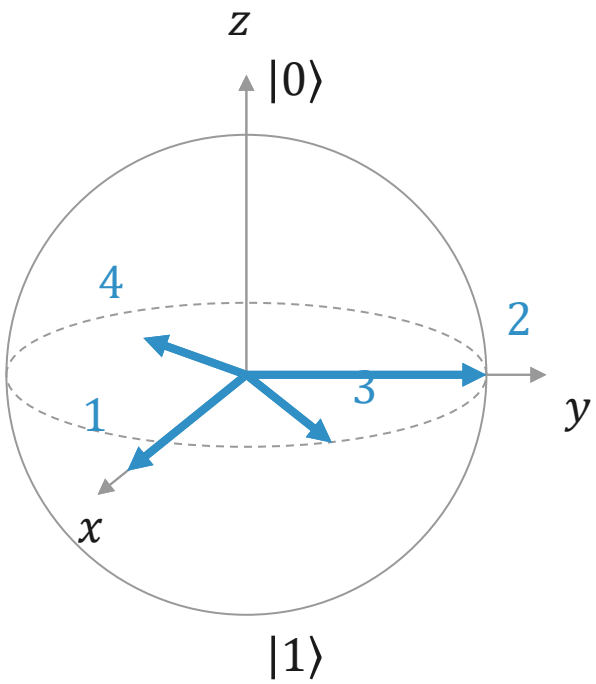
Simulation: Bloch vectors of gate sequences

```
paulis = [SparsePauliOp(p) for p in "XYZ"]
cases = {"H|0>": ["h"], "SH|0>": ["h", "s"],
  "TH|0>": ["h", "t"], "Ry(pi/3)|0>": ["ry"]}
qc = QuantumCircuit(1)
getattr(qc, g)(0)
sv = Statevector(qc)
```

Bloch vector = expectation values of X, Y, Z (Lecture 5).

Gates are applied in time order: "h", "t" means TH.

Sphere: 1 $H|0\rangle$ 2 $SH|0\rangle$ 3 $TH|0\rangle$ 4 $\mathrm{R}_y\left(\frac{\pi}{3}\right)|0\rangle$



TH|0⟩: r = (+0.7071, +0.7071, +0.0000)

SIMULATION

Simulation: the identities, numerically

```
T = np.diag([1, np.exp(1j * np.pi / 4)])
def R(sigma, theta):
"HZH = X": (H @ Z @ H, X),
"Rz(pi) = -iZ": (R(Z, np.pi), -1j * Z),
"Rz(pi) = Z": (R(Z, np.pi), Z),
for name, (a, b) in checks.items():
```

R(σ, θ) = cos(θ/2) I − i sin(θ/2) σ; SciPy's expm checks it.

np.allclose compares entries, so it sees global phases.

identity	allclose
HXH = Z	True
HZH = X	True
X^2 = I	True
S^2 = Z	True
T^2 = S	True
Rz(pi) = -iZ	True
Rz(pi) = Z	**False**
P(phi) = e^(i phi/2) Rz(phi)	True
Rx(theta) = expm(-i theta X/2)	True

Recap

1

Gates

X, Y, Z, H Hermitian; $S = \mathrm{P}(\pi/2), T = \mathrm{P}(\pi/4)$

2

Bloch sphere

angles θ, φ; orthogonal states are antipodal

3

Rotations

$\mathrm{R}_z(\theta) = e^{-i\theta\, Z/2}$; gates are rotations

4

Any gate

$e^{i\alpha}\, \mathrm{R}_z\, \mathrm{R}_y\, \mathrm{R}_z$

Next: Lecture 5

Dynamics and observables

Read: handout 2.5–2.7; then 2.8–2.9 before Lecture 5.

Exit ticket

Write $\mathrm{R}_z\left(\frac{\pi}{2}\right)$ as a matrix and compare it with S. What is the global phase between them?

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