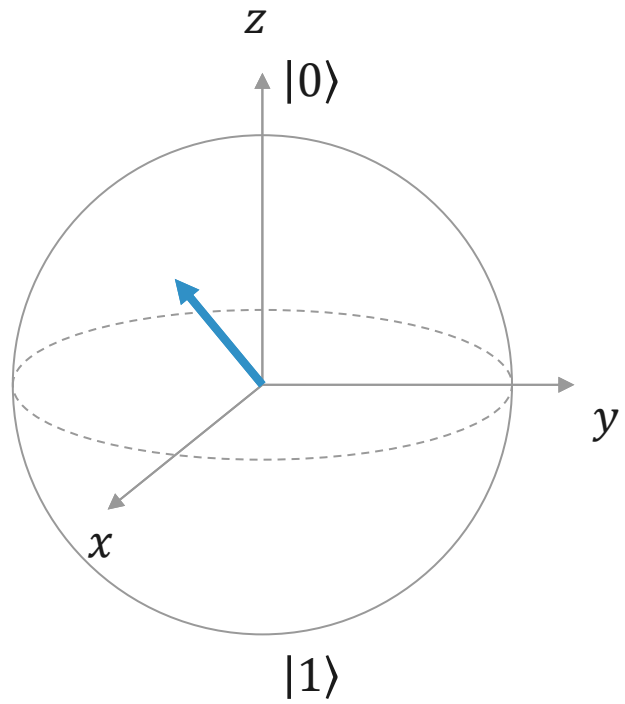
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

States and Measurement

The postulates of quantum mechanics for one qubit



Chapter 2 · Lecture 3

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Today



By the end of this lecture you can

- state the state, measurement and evolution postulates for one qubit
- compute outcome probabilities and post-measurement states in any orthonormal basis
- explain why a global phase is unobservable and a relative phase is not
- predict measurement counts and their binomial spread, and check them by simulation

YOUR TURN · RECALL

Recap: Hadamard twice

Recall · 45 seconds

Using $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, what is $H(H|0\rangle)$?

A $|0\rangle$

B $|1\rangle$

C $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

D equal amplitudes, as for two random flips

ANSWER

A: the amplitudes of |1⟩ cancel

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

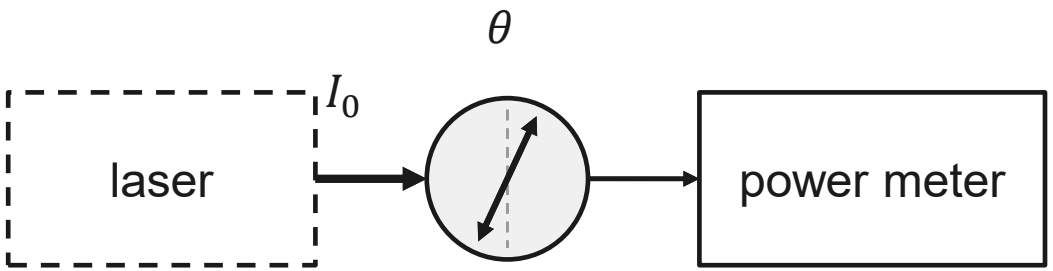
$$H(H|0\rangle) = \left(\frac{1}{2} + \frac{1}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{1}{2}\right)|1\rangle = |0\rangle$$

Today's question: what would a measurement show after one H, and after two?

1 Polarisers and single photons

What does an experiment actually record?

A polarised beam: Malus's law



$$I = I_0 \cos^2\theta$$

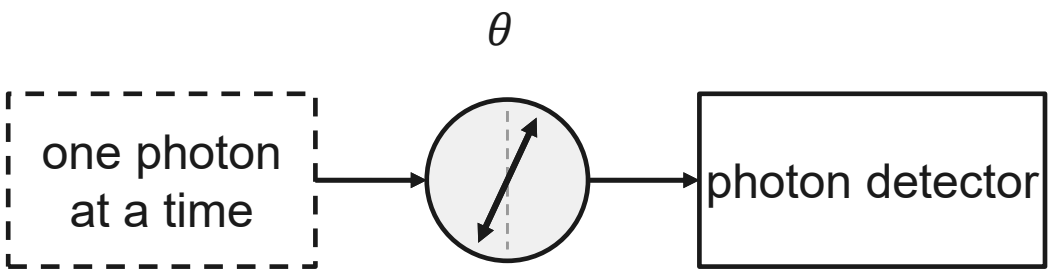
steady reading

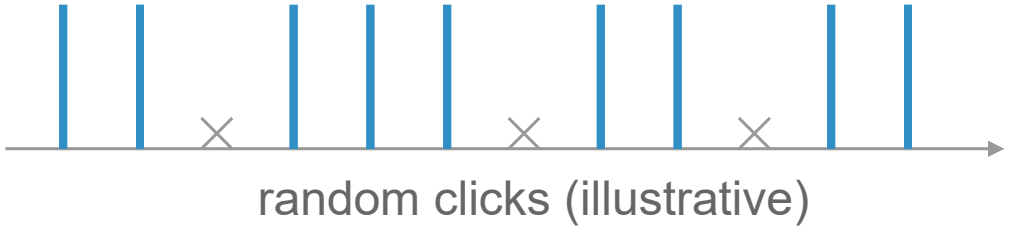
The field component along the axis passes: amplitude × cos θ.

Intensity ∝ amplitude², so the meter reads a steady fraction.

Every EE can predict this. Now dim the source.

One photon at a time: clicks





Each photon gives a whole click or nothing.

Which one happens is random.

Only the fraction of clicks approaches $\cos^2\theta$ (LaPierre §6.8).

EE LINK

Two lessons from the polariser

Lesson 1 · the state is a unit vector

$$|\psi\rangle = \cos\theta\,|V\rangle + \sin\theta\,|H\rangle$$

Lesson 2 · probability from an inner product

$$\Pr(\text{yes}) = |\langle V|\psi\rangle|^2 = \cos^2\theta$$

Malus's law is the average of many random answers.

The wave picture cannot supply the discreteness or the randomness.

Quantum mechanics predicts only the probability of each event.

2 The state postulate

What is the state of one qubit?

A qubit is a unit vector in $\mathbb{C}^2$

Principle · state postulate (one qubit)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

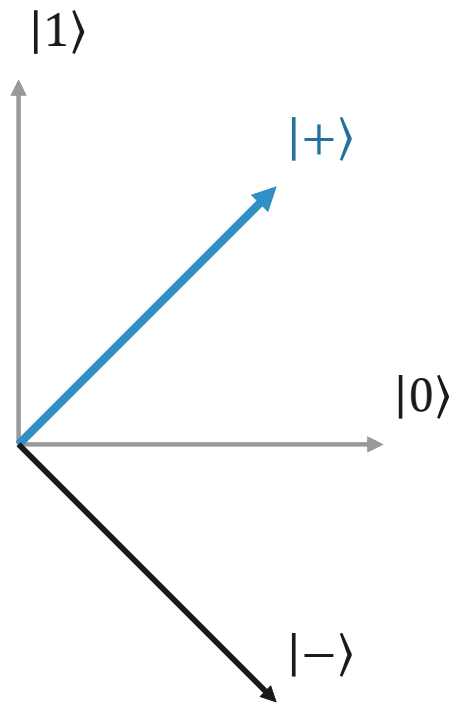
$$\alpha, \beta \in \mathbb{C}, \qquad |\alpha|^2 + |\beta|^2 = 1$$

Any system with two distinguishable configurations: polarisation, two energy levels, spin.

α and β are the amplitudes of $|0\rangle$ and $|1\rangle$.

The postulates do not depend on the hardware.

Superposition describes a vector, not a mixture



$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is one definite state.

It is a superposition in the computational basis …

… and a basis vector of the ± basis.

'Superposition' always refers to a chosen basis.

COMMON MISTAKE

“A qubit in superposition is both 0 and 1”

✗ Two values stored at once

The qubit holds 0 and 1 together.

A measurement reveals both.

A measurement of $|+\rangle$ returns one bit.

✓ Amplitudes that can interfere

Measuring $|+\rangle$ gives 0 or 1, never both.

The amplitudes are never read out.

They can cancel (two Hadamards).

What is new is interference, not storage.

One qubit, one measurement, one bit. The amplitudes decide the probabilities and can interfere.

3 Measurement

How do we get a classical answer out of a qubit?

The measurement postulate

Principle · measurement (orthonormal basis)

$$\Pr(k) = |\langle e_k|\psi\rangle|^2 \quad \text{(Born rule)}$$

$$\text{after outcome } k\text{:} \quad |\psi\rangle \to |e_k\rangle$$

Computational basis

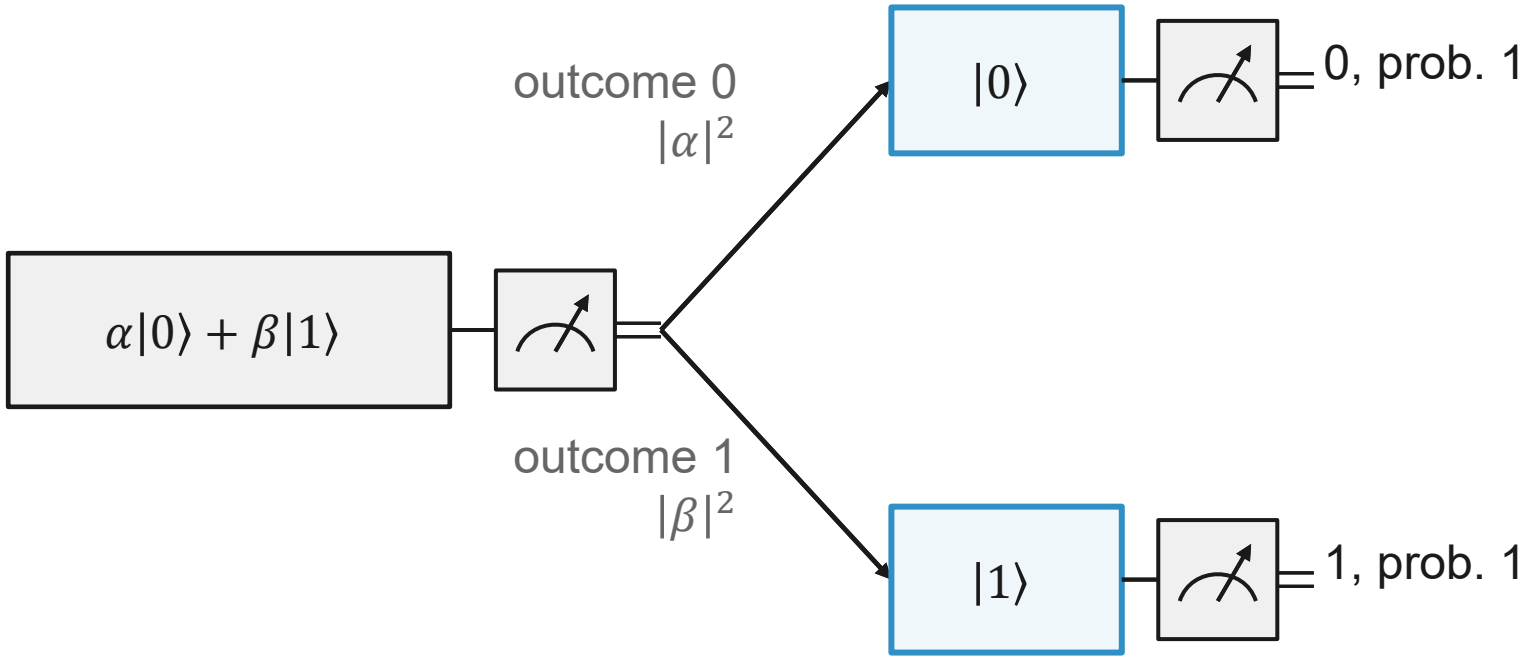
$$\Pr(0) = |\alpha|^2, \qquad \Pr(1) = |\beta|^2$$

The outcome is a classical label; the post-measurement state is a ket.

Measurement changes the state: the amplitudes are lost.

Probabilities add to 1 in every orthonormal basis.

Outcome, post-measurement state, repetition



Repeating the measurement immediately repeats the outcome.

YOUR TURN · PREDICT

Your turn: $|0\rangle$ measured in the ± basis

Think · 1 minute, then vote

A qubit in $|0\rangle$ is measured in the basis $\{|+\rangle, |-\rangle\}$. What is $\Pr(+)$?

A 0

B $\frac{1}{2}$

C $\frac{1}{\sqrt{2}}$

D 1

Worked example: one state, two bases

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + (1+i)|1\rangle)$$

$$\Pr(0) = \frac{1}{3}, \qquad \Pr(1) = \frac{|1+i|^2}{3} = \frac{2}{3}$$

1 Check the norm; Z basis: squared moduli.

Before the next slide

Expand ⟨±|ψ⟩ with ⟨±|1⟩ = ±1/√2.

Worked example: one state, two bases

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + (1+i)|1\rangle)$$

$$\Pr(0) = \frac{1}{3}, \qquad \Pr(1) = \frac{|1+i|^2}{3} = \frac{2}{3}$$

$$\langle +|\psi\rangle = \frac{2+i}{\sqrt{6}}, \qquad \langle -|\psi\rangle = \frac{-i}{\sqrt{6}}$$

1 Check the norm; Z basis: squared moduli.

2 ± basis: take the inner products first.

Before the next slide

Square the moduli. Do they add to 1?

Worked example: one state, two bases

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + (1+i)|1\rangle)$$

$$\Pr(0) = \frac{1}{3}, \qquad \Pr(1) = \frac{|1+i|^2}{3} = \frac{2}{3}$$

$$\langle +|\psi\rangle = \frac{2+i}{\sqrt{6}}, \qquad \langle -|\psi\rangle = \frac{-i}{\sqrt{6}}$$

$$\Pr(+) = \frac{5}{6} \approx 0.833, \qquad \Pr(-) = \frac{1}{6} \approx 0.167$$

1 Check the norm; Z basis: squared moduli.

2 ± basis: take the inner products first.

3 Same state, different statistics. $\mathrm{Im}\,\beta$ matters only in the ± basis.

4 Global and relative phase

Which parts of a state vector are physical?

A global phase changes nothing

Theorem · a global phase is unobservable

$|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$: same probabilities in every basis

Proof

$$\left|\langle e_k|e^{i\gamma}\psi\rangle\right|^2 = \left|e^{i\gamma}\right|^2|\langle e_k|\psi\rangle|^2 = |\langle e_k|\psi\rangle|^2$$

A gate before the meter does not help: $Ue^{i\gamma}|\psi\rangle = e^{i\gamma}U|\psi\rangle$.

$|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$ are the same physical state.

A relative phase is observable

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)$$

Z basis: $\frac{1}{2}, \frac{1}{2}$ for every φ

Other bases: depend on φ

In a.c. circuits only phase differences between phasors matter.

A global phase is like re-choosing the time origin.

A relative phase decides whether contributions reinforce or cancel.

YOUR TURN · DISCUSS

Your turn: which states can be told apart?

Pairs · 90 seconds

$|+\rangle$, $e^{i\pi/3}|+\rangle$, $|-\rangle$

Which pairs can some measurement distinguish? Which basis would you use?

ANSWER

Only the relative phase can be detected

Z basis: all three give $\frac{1}{2}, \frac{1}{2}$.

± basis: $|+\rangle$ and $e^{i\pi/3}|+\rangle$ give + with certainty; $|-\rangle$ gives − with certainty.

$|+\rangle$ vs $e^{i\pi/3}|+\rangle$: global phase, never distinguishable.

$|+\rangle$ and $|-\rangle$ differ by a relative phase of π: a ± measurement separates them.

5 Gates are unitaries

What happens between preparation and measurement?

Gates are unitary matrices

Principle · evolution (discrete form)

$$|\psi'\rangle = U|\psi\rangle, \qquad U^\dagger U = I$$

First gates

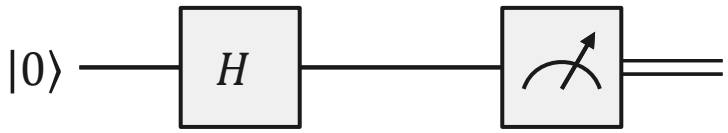
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

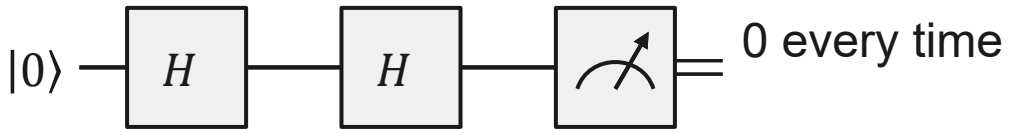
X: NOT gate, $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$

Z: phase flip, $Z|+\rangle = |-\rangle$

Unitarity keeps the total probability equal to 1.

Gate, then measure: the puzzle resolved





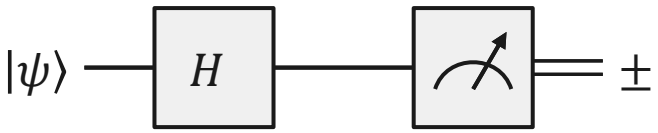
paths to $|1\rangle$: $+\frac{1}{2}$ and $-\frac{1}{2}$ cancel

One H: outcomes like a fair coin.

Two random flips leave a bit random; two Hadamards undo each other.

This cancellation is interference: no probabilistic counterpart.

A gate before the meter changes the basis



$$\Pr(k) = |\langle k|H|\psi\rangle|^2 = |\langle e_k|\psi\rangle|^2$$

$$|e_0\rangle = H|0\rangle = |+\rangle, \qquad |e_1\rangle = H|1\rangle = |-\rangle$$

Processors measure in the computational basis only.

Apply $U^\dagger$, then measure: a measurement in the basis $\{U|k\rangle\}$.

Circuit order A then B is the matrix product BA.

6 Outcomes, shots and counts

How many repetitions does an estimate need?

Shots and counts

Binomial spread of the count of 0

$$\sigma = \sqrt{Np(1-p)}$$

$$N = 1000, \qquad p = \cos^2 \frac{\pi}{8}$$

$$Np = 853.6, \qquad \sigma = 11.2$$

Shot: one preparation and measurement.

Counts: the histogram of outcomes.

1–2 σ from Np is normal; 5 σ means something is wrong.

YOUR TURN · PREDICT

Predict, then simulate: 1000 shots

Predict · 45 seconds

1000 measurements of $\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$. Which count of zeros is plausible?

A exactly 854, every run

B about 820 to 890

C about 450 to 550

D 990 or more

ANSWER

B: the count scatters by about ±11 around 854

Expected $Np = 853.6$, spread $\sigma = 11.2$: nearly every run lands within $\pm 3\sigma$.

No single run is guaranteed to give 854; a different seed gives a different count.

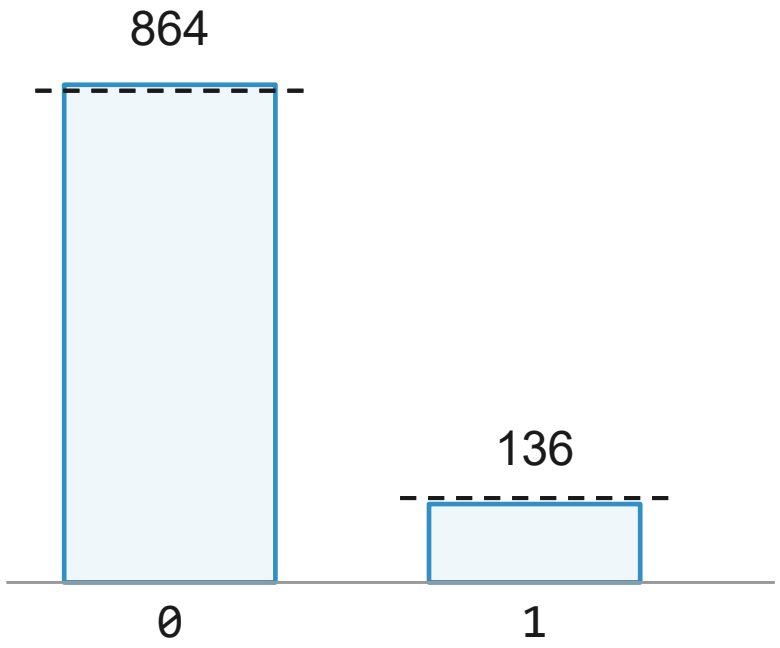
SIMULATION

Simulation: sampling the Born rule

```
psi = np.array([np.cos(a), np.sin(a)])
p = np.abs(psi) ** 2
rng = np.random.default_rng(6517)
outcomes = rng.choice([0, 1], size=shots, p=p)
n0 = int(np.sum(outcomes == 0))
sigma = np.sqrt(shots * p[0] * (1 - p[0]))
```

Born rule: $p = |\alpha|^2, |\beta|^2$ = 0.8536, 0.1464

Seeded generator (6517); 1000 shots.



count of 0 = 864: +0.93 σ from 853.6

SIMULATION

Simulation: global versus relative phase

```
sim = AerSimulator(seed_simulator=6518)
qc = QuantumCircuit(1, 1)
qc.prepare_state(amps, 0)
if basis == "X":
    qc.h(0)
p0 = Statevector(qc).probabilities()[0]
qc.measure(0, 0)
```

X row: H before the meter (± basis). 2000 shots per row.

Same seed, same distribution: identical counts in the Z rows.

state	basis	P(0)	counts 0/1
$\|+\rangle$	Z	0.500	996/1004
$\|+\rangle$	X	1.000	2000/0
$e^{i\pi/3}\|+\rangle$	Z	0.500	996/1004
$e^{i\pi/3}\|+\rangle$	X	1.000	2000/0
$\|-\rangle$	Z	0.500	996/1004
$\|-\rangle$	X	0.000	0/2000

Recap

1

State

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, |\alpha|^2 + |\beta|^2 = 1$

2

Measurement

$\Pr(k) = |\langle e_k|\psi\rangle|^2$; the state becomes $|e_k\rangle$

3

Phase

global: unobservable; relative: visible after a basis change

4

Gates and shots

unitaries; counts scatter by $\sqrt{Np(1-p)}$

Next: Lecture 4

Single-qubit gates and the Bloch sphere

Read: handout 2.1–2.5; then 2.6–2.7 before Lecture 4.

Exit ticket

Can $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ be distinguished by a Z measurement? By some measurement?

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