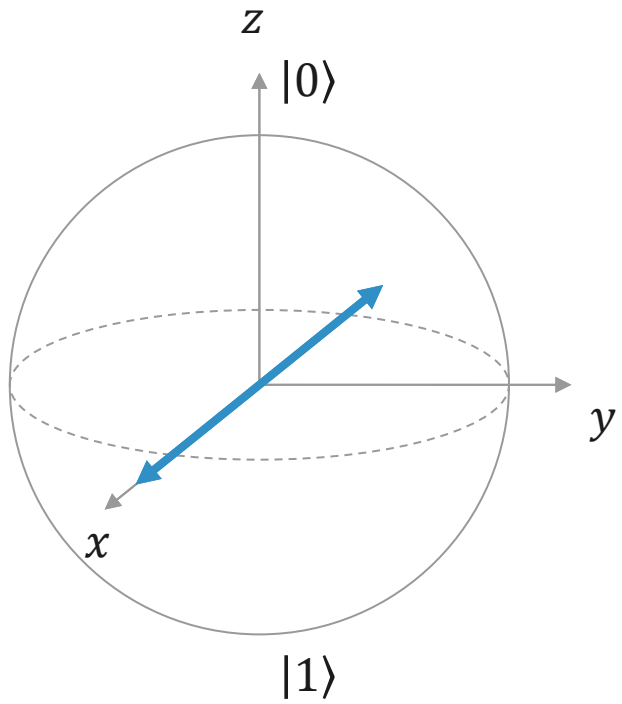
SAJID Lab
Smart & Advanced Junction of Intelligent Devices

EEE 6516 · QUANTUM COMPUTING

Inner Products, Operators and Eigenvalues



The linear algebra of quantum states

Chapter 1 · Lecture 2

Sajid Muhaimin Choudhury

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Today

1. **Inner products**
2. **Bases**
3. **Operators**
4. **Hermitian and unitary**
5. **Eigenvalues**
6. **Tensor preview**

By the end of this lecture you can

- compute inner products and norms; expand a state in an orthonormal basis
- build operators from outer products; use projectors and completeness
- decide whether a matrix is Hermitian, unitary, both or neither
- find eigenvalues and spectral decompositions; compute $e^{-i\theta\, X/2}$

YOUR TURN · RECALL

Recap: which is the bra?

Recall · 45 seconds

Which row vector is $\langle\psi|$ for $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$?

A $[1 \quad -i]/\sqrt{2}$

B $[1 \quad i]/\sqrt{2}$

C the column $(1, i)^T/\sqrt{2}$

D $[1 \quad i]/2$

ANSWER

B: conjugate and transpose

$$\langle\psi| = \frac{1}{\sqrt{2}}[1 \quad i]$$

The $-i$ becomes $+i$; the row is a conjugate transpose.

1 Inner products and bases

How do we measure the overlap of two states?

The inner product: bra times ket

Definition · inner product

$$\langle \varphi | \psi \rangle = {\varphi_0}^* \psi_0 + {\varphi_1}^* \psi_1$$

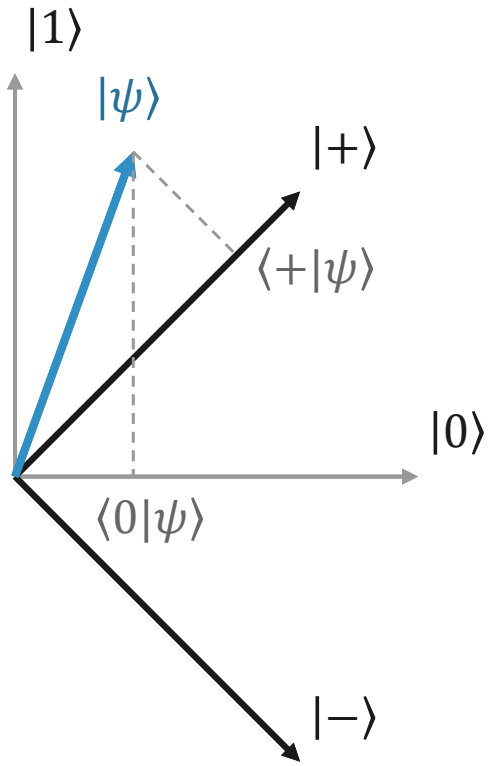
$$\|\psi\| = \sqrt{\langle \psi | \psi \rangle},$$
$$\langle \varphi | \psi \rangle = 0 \iff \text{orthogonal}$$

The conjugate makes $\langle \psi | \psi \rangle = |\psi_0|^2 + |\psi_1|^2$ real and non-negative.

Order matters: $\langle \psi | \varphi \rangle = \langle \varphi | \psi \rangle^*$

Orthonormal basis: unit length, mutually orthogonal, spanning the space.

Coefficients are inner products



$$|\psi\rangle = c_0|e_0\rangle + c_1|e_1\rangle, \qquad c_i = \langle e_i|\psi\rangle$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

The picture holds for real amplitudes only.

EE LINK

Inner products as signal projections

Fourier analysis: each coefficient is the inner product of the signal with one basis function.

$c_i = \langle e_i | \psi \rangle$ is the same idea in two dimensions.

Differences: finite dimension here, and the conjugate always sits on the bra.

WORKED EXAMPLE · STEP 1 OF 2

Worked example: the ± basis is orthonormal

$$\langle +| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

1 Real coefficients: the bra of $|+\rangle$ needs no conjugation.

Before the next slide

Expand $\langle +|-\rangle$ term by term.

Worked example: the ± basis is orthonormal

$$\langle +| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

$$\langle +|\,-\rangle = \frac{1}{2}(1 - 0 + 0 - 1) = 0$$

$$\langle 0|\,+\rangle = \frac{1}{\sqrt{2}} \approx 0.7071$$

1 Real coefficients: the bra of $|+\rangle$ needs no conjugation.

2 Use $\langle 0|0\rangle = \langle 1|1\rangle = 1, \langle 0|1\rangle = 0$. Similarly $\langle +|\,+\rangle = 1$.

YOUR TURN · PREDICT

Your turn: $|0\rangle$ in the ± basis

Think · 1 minute, then vote

Write $|0\rangle = c_+|+\rangle + c_-|-\rangle$. What is c_+?

A $\frac{1}{2}$

B $\frac{1}{\sqrt{2}}$

C 1

D 0

ANSWER

B: the coefficient is 1/√2

$$c_+ = \langle +|0\rangle = \frac{1}{\sqrt{2}}, \qquad |0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Its squared modulus is $\frac{1}{2}$: coefficient and probability are different numbers.

In general $\alpha|0\rangle + \beta|1\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$

Coefficients are inner products; probabilities (Chapter 2) are their squared moduli.

SIMULATION

Inner products in NumPy

```
psi = np.array([3, 4j]) / 5
def bra(ket):
    return ket.conj()
def inner(phi, psi):
    return bra(phi) @ psi
```

$|\psi\rangle = \frac{1}{5}(3|0\rangle + 4i|1\rangle)$: swapping the order conjugates.

np.vdot conjugates its first argument, exactly like a bra.

```
<psi|psi> = 1.0
<+|->     = 0.0
<+|psi>   = (0.4243+0.5657j)
<psi|+>   = (0.4243-0.5657j)
```

2 Operators as matrices

Two ways to read a 2 × 2 matrix.

Matrix elements and outer products

Matrix elements

$$A_{ij} = \langle i|A|j\rangle, \qquad \langle 0|X|1\rangle = 1$$

Outer products

$$A = \sum_{i,j} \langle i|A|j\rangle \, |i\rangle\langle j|, \qquad X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

A ket times a bra, $|\varphi\rangle\langle\chi|$, is a column times a row: a matrix.

Projectors and completeness

Definition · projector

$$P = |\psi\rangle\langle\psi|, \qquad P^2 = P = P^\dagger$$

$$P|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$$

Completeness

$$\sum_i |e_i\rangle\langle e_i| = I$$

P keeps only the component along its state.

Inserting $I = |e_0\rangle\langle e_0| + |e_1\rangle\langle e_1|$ in front of a ket gives its expansion in that basis.

3 Hermitian and unitary matrices

Two classes the whole course depends on.

Hermitian and unitary

Definition · adjoint, Hermitian, unitary

$$(A^\dagger)_{ij} = A_{ji}^*$$

Hermitian: $A^\dagger = A$ unitary: $U^\dagger U = I$

Theorem · unitaries preserve inner products

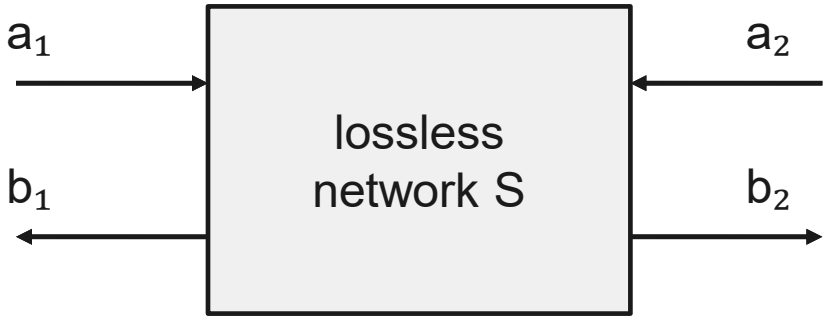
$$\langle \varphi | U^\dagger U | \psi \rangle = \langle \varphi | \psi \rangle$$

Hermitian matrices will describe measurable quantities.

Unitary matrices will describe what a quantum computer does.

Proof: the bra of $U|\varphi\rangle$ is $\langle \varphi | U^\dagger$.

A lossless network is unitary



$$b = S\,a, \qquad S^{\dagger}\,S = I$$

power in = power out

Waves normalised so $|a_i|^2$ is a power: lossless means $S^{\dagger}\,S = I$ (Pozar).

A qubit unitary conserves $|\alpha|^2 + |\beta|^2$ the same way.

Same algebra; the network conserves power, the unitary conserves total probability.

YOUR TURN · CLASSIFY

Your turn: Hermitian, unitary, both or neither?

Pairs · 2 minutes

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Classify each matrix.

ANSWER

Only U†U = I decides unitarity

X, H: both. S: unitary only.

B: Hermitian only, $B^\dagger B = 2B \neq I$.

N: neither, $N^\dagger N = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq I$.

Real entries, symmetry or non-negative entries do not make a matrix unitary.

COMMON MISTAKE

“Real and symmetric, so it must be unitary”

✗ Judging by appearance

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

real, symmetric, non-negative

B is Hermitian, but it is not unitary.

✓ Test the definition

$$B^\dagger B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2B$$

$$2B \neq I$$

Compute U†U and compare with I.

Hermitian and unitary are different properties: X and H have both, S and B only one each.

4 Eigenvalues and the spectral theorem

Why Hermitian and unitary matrices behave so well.

Eigenvalues of Hermitian and unitary matrices

Theorem · Hermitian eigenvalues

$$A = A^\dagger \;\Rightarrow\; \lambda \text{ real};\quad \lambda \neq \mu \;\Rightarrow\; \langle w|v\rangle = 0$$

Spectral theorem

$$A = \sum_i \lambda_i |v_i\rangle\langle v_i|$$

Unitary eigenvalues

$$|\lambda| = 1, \qquad \lambda = e^{i\varphi}$$

Proof idea: $\langle v|A|v\rangle = \lambda\langle v|v\rangle$ and its conjugate give $\lambda = \lambda^*$.

Full eigenbasis: proved in the reference section (Nielsen & Chuang, Box 2.2).

Example: $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$ has eigenvalues 0 and 3.

WORKED EXAMPLE · STEP 1 OF 3

Worked example: a function of X

$$X|+\rangle = |+\rangle, \qquad X|-\rangle = -|-\rangle \;\Rightarrow\; X = |+\rangle\langle +| - |-\rangle\langle -|$$

1 Eigenvalues ±1, eigenvectors $|\pm\rangle$ (orthonormal).

Before the next slide

Apply the exponential to each eigenvalue.

Worked example: a function of X

$$X|+\rangle = |+\rangle, \qquad X|-\rangle = -|-\rangle \;\Rightarrow\; X = |+\rangle\langle+| - |-\rangle\langle-|$$

$$e^{-i\theta\, X/2} = e^{-i\theta/2}|+\rangle\langle+| + e^{i\theta/2}|-\rangle\langle-|$$

1 Eigenvalues ±1, eigenvectors $|\pm\rangle$ (orthonormal).

2 A function of a Hermitian matrix acts on its eigenvalues.

Before the next slide

Rewrite with cos and sin.

Worked example: a function of X

$$X|+\rangle = |+\rangle, \qquad X|-\rangle = -|-\rangle \;\Rightarrow\; X = |+\rangle\langle+| - |-\rangle\langle-|$$

$$e^{-i\theta\, X/2} = e^{-i\theta/2}|+\rangle\langle+| + e^{i\theta/2}|-\rangle\langle-|$$

$$e^{-i\theta\, X/2} = \cos\frac{\theta}{2}\, I - i\sin\frac{\theta}{2}\, X$$

1 Eigenvalues ±1, eigenvectors $|\pm\rangle$ (orthonormal).

2 A function of a Hermitian matrix acts on its eigenvalues.

3 Sum and difference of the projectors: I and X.

SIMULATION

Three routes to the same matrix

```
theta = np.pi / 3
U1 = expm(-1j * theta * X / 2)
lam, V = np.linalg.eigh(X)
print("expm = cos/sin :", np.allclose(U1, U3))
```

U2 rebuilds the matrix from the spectral decomposition; U3 is the cos/sin formula.

Unitary: this is the rotation gate $R_x(\pi/3)$ of Chapter 2.

```
[[0.866+0.j  0.   -0.5j]
 [0.   -0.5j 0.866+0.j ]]
expm = spectral: True
expm = cos/sin : True
unitary        : True
```

Preview: two qubits, four amplitudes

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

Each new qubit doubles the vector: the memory wall again.

Qubit 0 is written on the left (textbook order).

Qiskit uses the opposite order; Chapter 3 converts carefully.

Recap

1

Inner products

$\langle\varphi|\psi\rangle = \sum \varphi_i^* \psi_i$; coefficients $c_i = \langle e_i|\psi\rangle$

2

Projectors

$P^2 = P = P^\dagger, \sum |e_i\rangle\langle e_i| = I$

3

Hermitian vs unitary

$A^\dagger = A$ vs $U^\dagger U = I$; not the same.

4

Spectral theorem

$A = \sum \lambda_i |v_i\rangle\langle v_i|, f(A) = \sum f(\lambda_i)|v_i\rangle\langle v_i|$

Next: Lecture 3 (Chapter 2)

States, measurement and the Born rule

Read: handout 1.5–1.9 and self-study 1.10–1.11.

Exit ticket

Find the eigenvalues of $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and write $e^{-i\theta\, Z/2}$ as a diagonal matrix.

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